Entropic entanglement criteria in phase space

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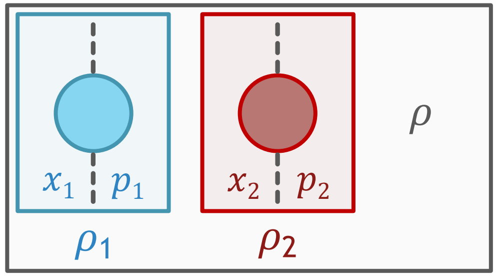
Big picture

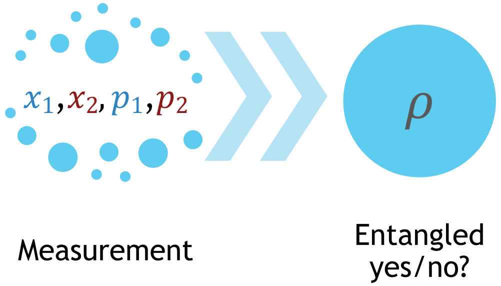
- Bipartition of two oscillator modes 1 and 2 with
  - Positions x_1, x_2 and momenta p_1, p_2
  - Global state ρ
  - Local states $\rho_1 = Tr_2\{\rho\}$, $\rho_2 = Tr_1\{\rho\}$



- Global state ρ is entangled if and only if

  $\rho \neq \sum_i p_i \,(\rho_1^i \otimes \rho_2^i)$, where p_i is a probability distribution

- Goal: Measure x_1, x_2, p_1, p_2 and try to certify entanglement between 1 and 2



for *as many states as possible*

→ Entropic entanglement criteria

→ „Joint“ measurements

Outline

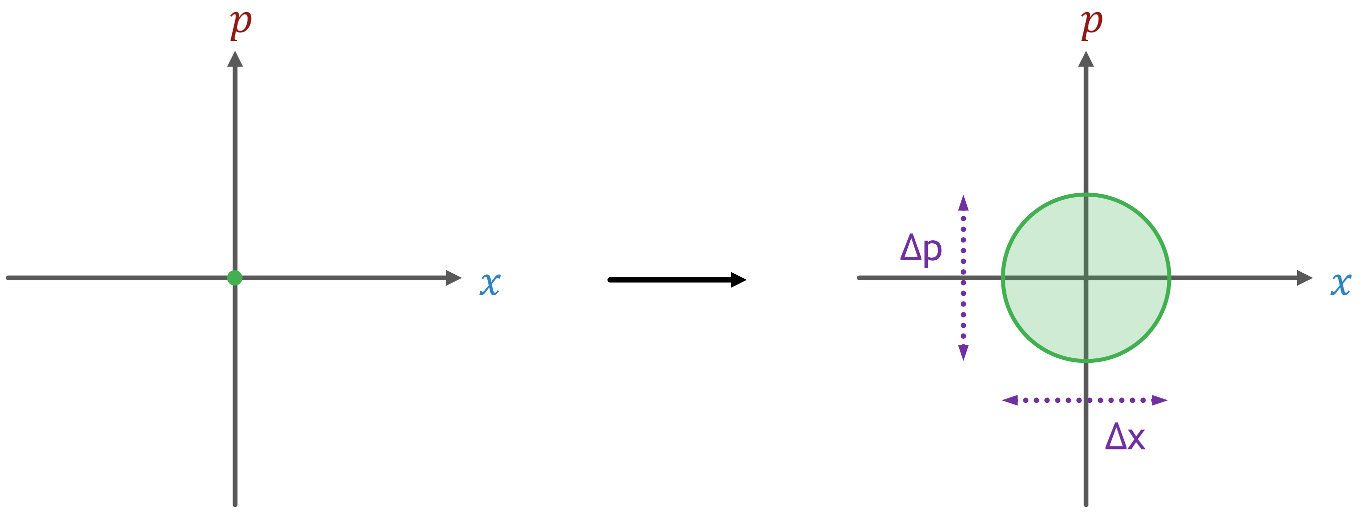
- Continuous variable systems and phase space
- Wehrl mutual information
- Inseperability criteria
- Comparison
- Stronger bounds and discretization (work in progress)

Continuous variable systems and phase space

- Monopartite system, start from a set of conjugate variables

$$[X, P] = i$$

- State has a representation in phase space, respect uncertainty principle

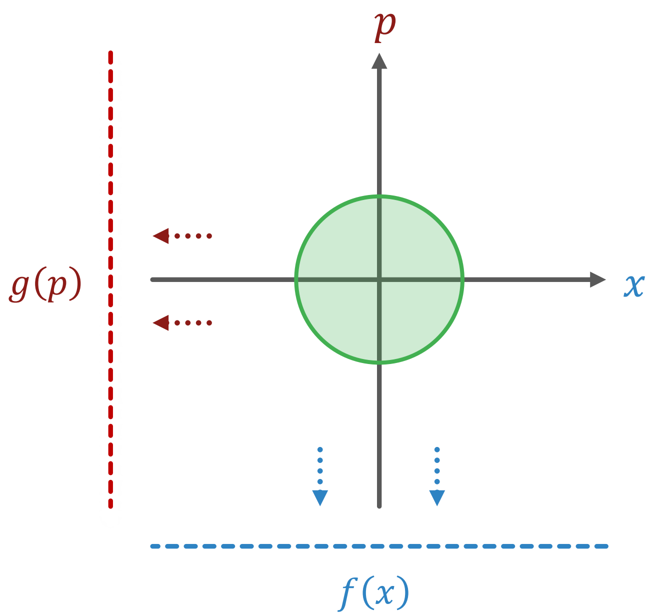


Classical mechanics: Coordinate

Quantum mechanics: Quasi-probability distribution(s)

Continuous variable systems and phase space

- Extract information in phase space by measuring position and momentum
- Homodyne detection
  - Project onto eigenstates $|x\rangle, |p\rangle$
  - Gives marginals $f(x) = \langle x|\rho|x\rangle, g(p) = \langle p|\rho|p\rangle$
  - Many (entropic) criteria exist Walborn *et al.* ´09
- Heterodyne detection
  - Project onto pure coherent states $|\alpha\rangle$
  - $a|\alpha\rangle = \alpha|\alpha\rangle$ with $\alpha = (x + i\, p)/\sqrt{2}$
  - Gives Husimi Q-distribution $\mathrm{Q}(x,p) = \langle\alpha|\rho|\alpha\rangle$
  - „Joint“ measurement of position and momentum

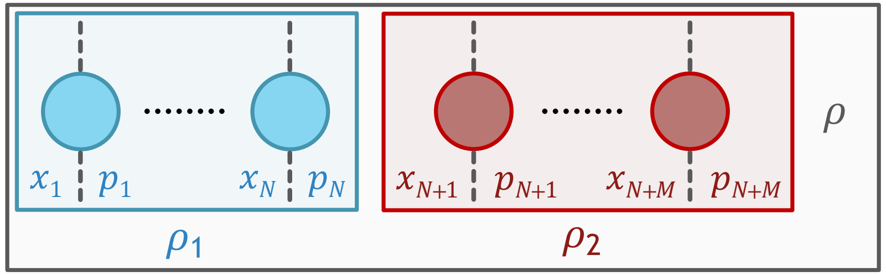


Continuous variable systems and phase space

- Husimi Q-distribution is
  - non-negative and bounded $0 \leq \mathrm{Q}(x, p) \leq 1$
  - normalized $\int \frac{dx\, dp}{2\pi} \mathrm{Q}(x, p) = 1$
  - a *quasi*-probability distribution as coherent states overlap
- We can associate an entropy to it, which is the Wehrl entropy[Wehrl´78´79]. It
  - is defined as $S_W(\mathrm{Q}) = -\int \frac{dx\, dp}{2\pi} \mathrm{Q}(x, p) \ln \mathrm{Q}(x, p)$
  - is a coarse-grained entropy $S_W(\mathrm{Q}) > S(\rho)$
  - fulfills an entropic uncertainty relation (Wehrl-Lieb inequality) $S_W(\mathrm{Q}) \geq N$, tight for all pure coherent states[Lieb´78, Lieb and Solovej´14]

Wehrl mutual information

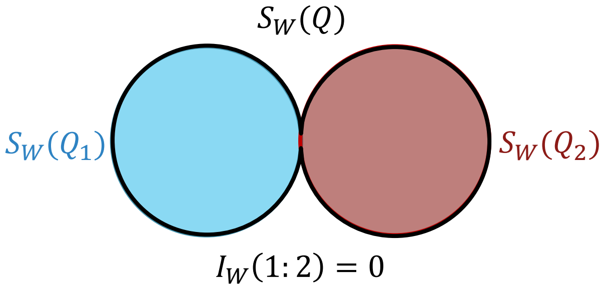
- General setup: bipartite system 1 and 2 with $N + M$ modes

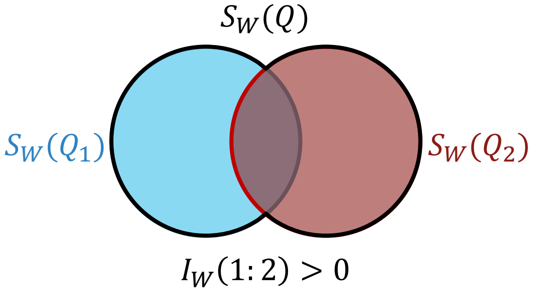


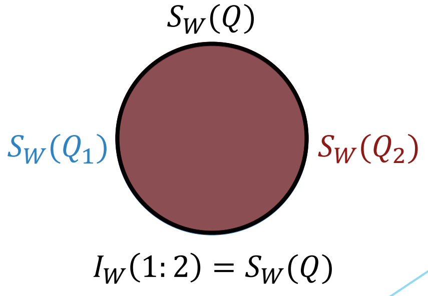
- To capture correlations in phase space, we define Wehrl mutual information

$$I_W(1{:}2) \equiv S_W(Q||Q_1 \times Q_2) = S_W(Q_1) + S_W(Q_2) - S_W(Q)$$

- Intuition for all involved quantities with Venn diagrams







Wehrl mutual information

- If global state ρ is pure, $I_W(1{:}2) = 0$ is a necessary and sufficient condition for separability

  → Measurable analog of the positive partial transpose (PPT) criterion for pure states and $N + M$ modes

- All pure entangled states are witnessed, e.g. N00N states for all N
- $I_W(1{:}2)$ is neither an entanglement measure, nor an entanglement monotone
- But, it is a lower bound on the entanglement entropy$^{\text{Lieb and Seiringer´05}}$

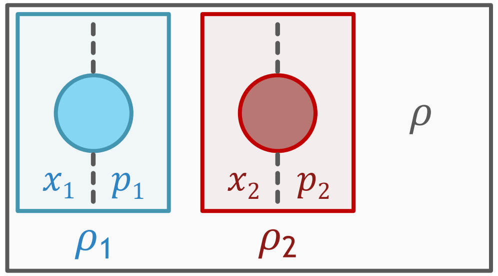
$$I_W(1{:}2) \leq \frac{1}{2} S(\rho_1) = \frac{1}{2} S(\rho_2)$$

  → „How much are 1 and 2 entangled *at least*?“

- If global state ρ is mixed, $I_W(1{:}2)$ also includes classical correlations ✗

Inseparability criteria

- To derive general inseparability criteria, we restrict to two modes 1 and 2



  with commutation relations $[X_j, P_k] = i\,\delta_{jk}\ \ (j,k = 1{,}2)$

- We allow for rotations in both phase spaces by introducing local angles ϑ_1, ϑ_2

$$\begin{pmatrix} R_j \\ S_j \end{pmatrix} = \begin{pmatrix} \cos\vartheta_j & \sin\vartheta_j \\ -\sin\vartheta_j & \cos\vartheta_j \end{pmatrix} \begin{pmatrix} X_j \\ P_j \end{pmatrix}$$

- Starting point: Global Husimi $Q(r_1, s_1, r_2, s_2)$
- Note: No angle tomography needed!

Inseparability criteria

- Transform to EPR-type variables

$$r_\pm = r_1 \pm r_2,\; s_\pm = s_1 \pm s_2$$

and consider the two mixed marginal distributions

$$Q_\pm(r_\pm, s_\mp) = \int \frac{dr\, ds}{2\pi} Q(r, s, \mp r \pm r_\pm, \pm s \mp s_\mp)$$

- Note that $[R_\pm, S_\mp] = 0 \rightarrow$ Marginals are not „true“ Husimi Q-distributions, but still bounded and normalized to 1
- For pure separable states $\rho = \rho_1 \otimes \rho_2$, the global Husimi Q-distribution factorizes

$$Q(r_1, s_1, r_2, s_2) = Q_1(r_1, s_1) \times Q_2(r_2, s_2)$$

Inseparability criteria

- Mixed marginals become

$$Q_\pm(r_\pm, s_\mp) = (Q_1 * Q_2^{(\pm)})(r_\pm, s_\mp)$$

with $Q_2^{(\pm)} = Q_2(\pm r, \mp s)$

- Use 2D entropy power inequality together with invariance of entropies under mirror reflections and the Wehrl-Lieb inequality

$$S(Q_\pm) \geq \ln(\underset{\substack{\uparrow \\ \text{strong criteria}}}{e^{S_W(Q_1)} + e^{S_W(Q_2)}}) \geq 1 \underset{\substack{\uparrow \\ \text{weak criteria}}}{+} \ln 2$$

- Generalize weak criteria to mixed separable states

$$\rho = \textstyle\sum_i p_i \, (\rho_1^i \otimes \rho_2^i) \;\rightarrow\; Q_\pm(r_\pm, s_\mp) = \textstyle\sum_i p_i \, Q_\pm^i(r_\pm, s_\mp)$$

Inseparability criteria

- Use concavity of entropies and weak criteria

$$S(Q_{\pm}) \geq \sum_i p_i\, S(Q_{\pm}^i) \geq \sum_i p_i\,(1+\ln 2) = 1+\ln 2$$

→ Weak criteria generalize to mixed separable states

- We end up with

$$\boxed{S(Q_{\pm}) \geq \ln\left(e^{S_W(Q_1)} + e^{S_W(Q_2)}\right)}$$

strong criteria
for pure separable states

$$\boxed{S(Q_{\pm}) \geq 1+\ln 2}$$

weak criteria
for mixed separable states

- Violation of these inequalities flags entanglement

Comparison

- Gaussians maximize entropies → we can infer second-order criteria

$$\left(\sigma_{r_\pm}^2 + a^2\right)\left(\sigma_{s_\mp}^2 + 1/a^2\right) \geq 4 + \sigma_{r_\pm s_\mp}^2,$$

  where a is the squeezing parameter

- Second-order criteria are invariant under rotations, but not under squeezing
- Complementary compared to MGVT criteria[MGVT´02] (implied by Walborn *et al.*)

$$\sigma_{r_\pm}\sigma_{s_\mp} \geq 1$$

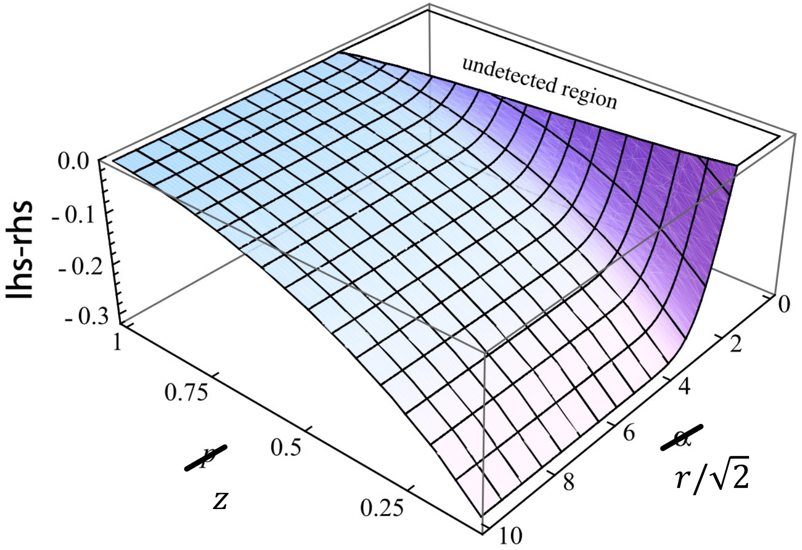
- After appropriate optimizations the two are equivalent
- Entropic criteria[Walborn *et al.* ´09] for homodyne marginals $f(r_1, r_2), g(s_1, s_2)$

$$S(f_\pm) + S(g_\mp) \geq \ln e\pi + \ln 2$$

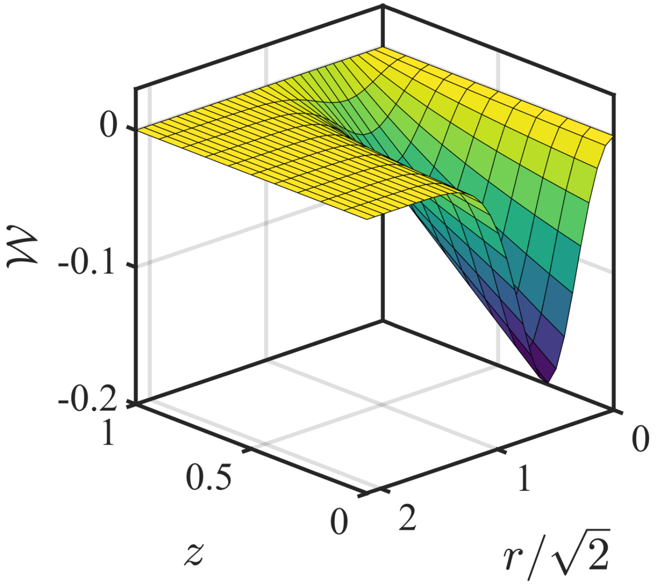
Comparison

- Dephased Schrödinger cat state, pure iff $z = 0$ and separable iff $\alpha = 0$ or $z = 1$

$$\rho = N(\alpha)\big(|\alpha,\alpha\rangle\langle\alpha,\alpha| + |-\alpha,-\alpha\rangle\langle-\alpha,-\alpha| + (1-z)(|\alpha,\alpha\rangle\langle-\alpha,-\alpha| + |-\alpha,-\alpha\rangle\langle\alpha,\alpha|)\big)$$



Picture taken from Walborn *et al.* ´09

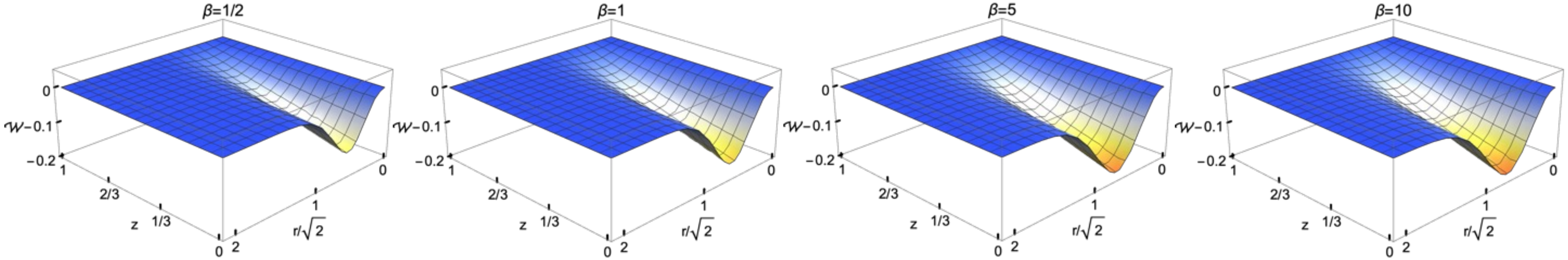


Stronger bounds and discretization

- Generalize to the family of Rényi-entropies $S^\beta(Q_\pm)$ with $\beta \neq 1$

$$S^\beta(Q_\pm) = \frac{1}{1-\beta}\ln\left(\int \frac{dr_\pm ds_\mp}{2\pi} Q_\pm^\beta\right) \geq \frac{\ln\beta}{\beta-1} + \ln 2$$

- Even stronger witness as β allows for optimization. Cat state from before:



- Discretize witness for application to experiments $Q_\pm \rightarrow q_\pm$

$$S^\beta(q_\pm) + \ln\frac{\Delta r_\pm \Delta s_\mp}{2\pi} \geq S^\beta(Q_\pm) \geq \frac{\ln\beta}{\beta-1} + \ln 2$$

Summary

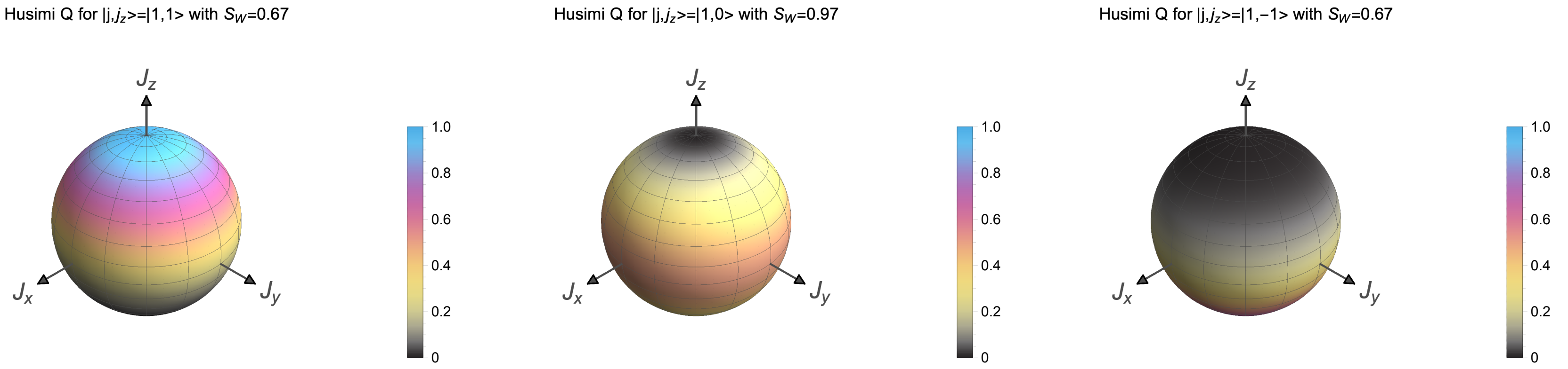
- For $N + M$ modes: Wehrl mutual information $I_W(1{:}2)$ is a measurable perfect witness for pure state entanglement and a lower bound on the entanglement entropy $S(\rho_1)$
- For $1 + 1$ modes: Inseparability criteria in terms of a Wehrl entropy

$$S(Q_\pm) \geq 1 + \ln 2$$

- Does not require angle tomography
- Fully witnesses the mixed cat state, works best for large overlaps
- Stronger witness for Rényi entropies $S^\beta(Q_\pm)$
- Can be applied to experiments via discretized distributions

Outlook

- Generalize to su(2) algebra: continuous Husimi Q, but compact phase space



Husimi Q for |j,j_z>=|1,1> with S_W=0.67

Husimi Q for |j,j_z>=|1,0> with S_W=0.97

Husimi Q for |j,j_z>=|1,−1> with S_W=0.67

or other groups → universal approach to *entropic* entanglement witnessing!

- Generalize Husimi Q and Wehrl entropy to quantum field theory

→ See also: Relative entropy formulation of entropic uncertainty for quantum fields, arXiv:2107.07824

Thanks for your attention!

References

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Backup slides

Comparison

- Husimi Q is related to Wigner W via a Weierstrass transform w.r.t. vacuum
- Husimi covariance matrix fulfills $V_{\pm} = \gamma_{\pm} + 1$ with $\gamma_{\pm} = \begin{pmatrix} \sigma_{r_{\pm}} & \sigma_{r_{\pm}s_{\mp}} \\ \sigma_{r_{\pm}s_{\mp}} & \sigma_{s_{\mp}} \end{pmatrix}$
- Behavior of second-order criteria under rotations (a) / squeezing (b)