Relative entropic uncertainty relation for scalar quantum fields

Stefan Floerchinger[1], Tobi Haas[1], Ben Hoeber[1], Markus Schröfl[1]

[1]Institut für Theoretische Physik

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UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

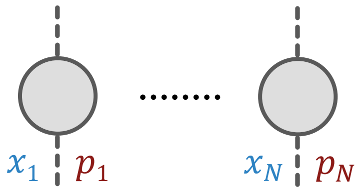
Big picture

- Consider N oscillator modes
  - N positions $\vec{x}$ and N momenta $\vec{p}$
  - Measuring them gives the distributions

    $$f(\vec{x}) = \langle x|\rho|x\rangle \text{ and } g(\vec{p}) = \langle p|\rho|p\rangle$$

  - Associated entropic uncertainty relation (EUR) reads

    $$S(f) + S(g) \geq N(1 + \ln \pi)$$



- Field theory limit $N \to \infty$ renders all quantities infinite ✗

$\rightarrow$ Use relative entropies $S(f||f_\alpha)$ instead

$\rightarrow$ Obtain finite results also in the field theory limit

Outline

- Entropic uncertainty relation for a single oscillator
- From oscillators to fields
- The relative entropic uncertainty relation
- Example: Excitations

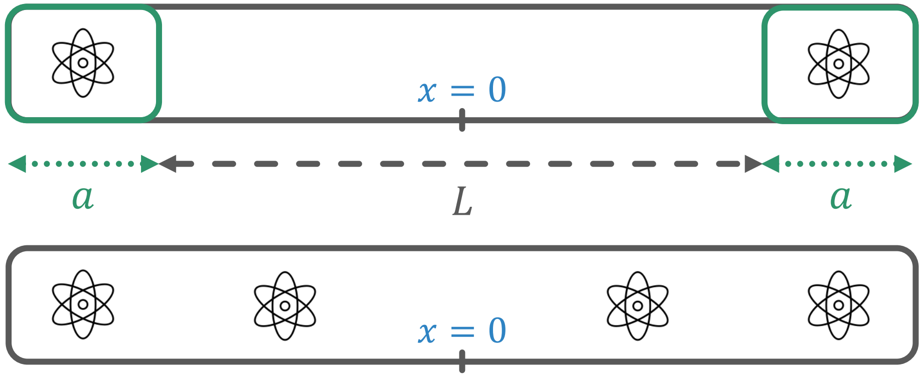
Flaws of standard deviation

- Let us start from Heisenberg‘s uncertainty relation

$$\sigma_x \sigma_p \geq \frac{1}{2}$$

- What is wrong with using the standard deviations σ_x and σ_p?[Coles et *al.* ’17]
  - No information about other moments of $f(x) = \langle x|\rho|x\rangle$ and $g(p) = \langle p|\rho|p\rangle$
  - Behave counterintuitively: Consider particle in boxes with $L \gg a$

1. With walls: $\sigma_x \approx 0.5\, L$

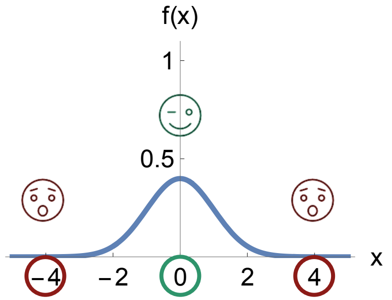


∨ ?!

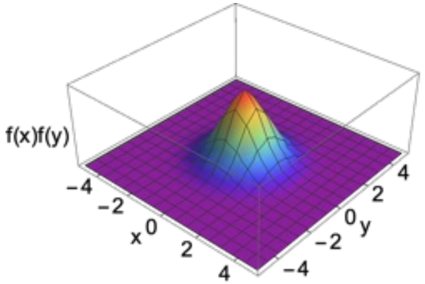
2. Without walls: $\sigma_x \approx 0.29\, L$

Measuring „surprise“

- Which properties should a good measure of uncertainty possess? Vedral ‘02
  - Think in terms of „surprise“: Given an event x with probability density $f(x)$, how surprised are we when the event *does* occur?



Unlikely events x with small $f(x)$ would surprise a lot $\rightarrow \frac{1}{f(x)}$



Surprisals of independent events x and y with probability density $f(x)\times f(y)$ should add up

$\rightarrow -\ln f(x)$

$\rightarrow -\ln\big(f(x)\times f(y)\big) = -\ln f(x) - \ln f(y)$

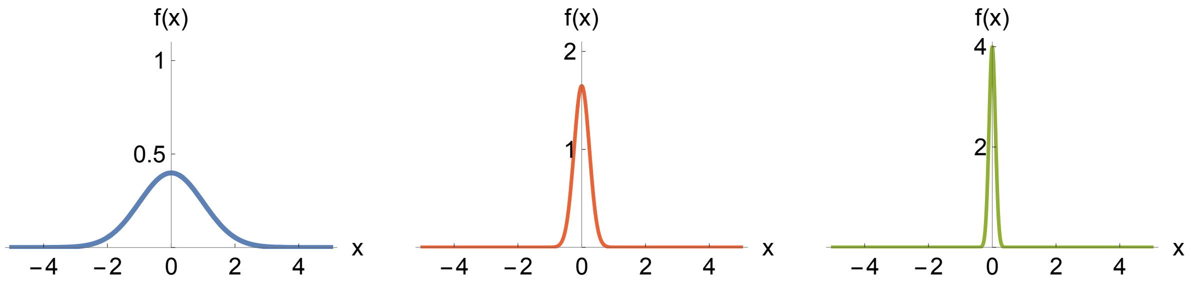
  - Surprise of event x is $-\ln f(x)$

Differential entropy

- We introduce differential entropy $S(f)$ as *the* measure of average surprise / uncertainty / missing information of a distribution $f(x)$

$$S(f) = -\int dx\, f(x) \ln f(x)$$

- Intuition: $S(f)$ is small whenever the distribution $f(x)$ is highly localized



$S(f) = 1.42 \quad > \quad S(f) = 0 \quad > \quad S(f) = -0.88$

$\rightarrow$ For continuous variables, e.g. position x or momentum p, entropies can be negative

$\rightarrow$ What happens when considering position x *and* momentum p?

BBM Entropic uncertainty relation

- Entropic uncertainty relation for position $f(x)$ and momentum $g(p)$ distributions by Białynicki-Birula and Mycielski (BBM)[Białynicki-Birula and Mycielski '75]

$$S(f) + S(g) \geq 1 + \ln \pi$$

$\rightarrow$ Either entropy can become small or even negative, but their sum is bounded from below by a positive number

- Stronger than Heisenberg's relation[Hertz and Cerf '19]
  - Q: Which distribution $f(x)$ maximizes entropy $S(f)$ for a given variance σ_x^2?
  - A: Gaussian $f(x) = f_G(x) \rightarrow S(f) \leq S(f_G) = \frac{1}{2}\ln(2\pi e \sigma_x^2)$
  - Hence, $\ln\left(2\pi e \sigma_x \sigma_p\right) \geq S(f) + S(g) \geq \ln \pi e \implies \sigma_x \sigma_p \geq \frac{1}{2}$
- Applications: entanglement witnesses[Walborn et *al.* '09], steering[Walborn et *al.* '11], …

Coupled oscillators and field theory limit

- Toy model: Chain of N coupled modes in $d = 1 + 1$ dimensions

Collection of oscillators	Limits
$H = \frac{1}{2}\sum_{j=0}^{N-1} \varepsilon \left[\pi_j^2 + \frac{1}{\varepsilon^2}(\phi_j - \phi_{j-1})^2 + m^2\phi_j^2\right]$	Continuum limit: $\varepsilon \to 0,\ \ N \to \infty,\ \ L = N\varepsilon =$c. $H = \frac{1}{2}\int_0^L dx\, [\pi^2(x) + (\partial_x\phi(x))^2 + m^2\phi^2(x)]$
$H = \frac{1}{2}\sum_\ell \frac{\Delta k}{2\pi}[\pi_\ell^2 + \omega_\ell^2\phi_\ell^2]$	Infinite volume limit: $\Delta k \to 0,\ \ N \to \infty,\ \ \varepsilon = \frac{L}{N}=$c. $H = \frac{1}{2}\int_{-\frac{\pi}{\varepsilon}}^{+\frac{\pi}{\varepsilon}} \frac{dp}{2\pi}\ [\pi^2(p) + \omega^2(p)\phi^2(p)]$
$\omega_\ell = \sqrt{\frac{4}{\varepsilon^2}\sin^2\left(\frac{\Delta k \ell \varepsilon}{2}\right) + m^2}$	$\omega(p) = \sqrt{p^2 + m^2}$

- Field theory limit: continuum limit + infinite volume limit

Schrödinger picture for quantum fields

- Schrödinger picture: States are defined on constant time slices

- Expand field operators as $\Phi_\ell|\phi\rangle = \phi_\ell|\phi\rangle$ and $\Pi_\ell|\pi\rangle = \pi_\ell|\pi\rangle$
- In this basis, the density matrix elements read $\rho[\phi_+, \phi_-] = \langle\phi_+|\rho|\phi_-\rangle$

  $\rightarrow$ Functional probability density $F[\phi] = \rho[\phi, \phi] = \langle\phi|\rho|\phi\rangle$
- Expectation values can be obtained via functional integrals

$$\langle\mathcal{O}(\phi)\rangle = \int \mathcal{D}\phi \;\; \mathcal{O}(\phi)\, F[\phi]\,, \;\; \mathcal{D}\phi = \prod_\ell \int d\phi_\ell \sqrt{\frac{\Delta k}{2\pi}}$$

Vacuum and coherent states

- Vacuum wave functionals $\bar{\Psi}[\phi]$ and $\bar{\Psi}[\pi]$ follow from Schrödinger's equation, associated functional probability densities read

$$\bar{F}[\phi] = |\bar{\Psi}[\phi]|^2 = \bar{Z}_\phi^{\;-1} \exp\left[-\frac{1}{2}\sum_\ell \frac{\Delta k}{2\pi}\sum_m \frac{\Delta k}{2\pi}\, \phi_\ell\, \bar{\mathcal{M}}^{-1}_{\ell m}\, \phi_m\right],$$

$$\bar{G}[\pi] = |\bar{\Psi}[\pi]|^2 = \bar{Z}_\pi^{\;-1} \exp\left[-\frac{1}{2}\sum_\ell \frac{\Delta k}{2\pi}\sum_m \frac{\Delta k}{2\pi}\, \pi_\ell\, \bar{\mathcal{N}}^{-1}_{\ell m}\, \pi_m\right],$$

with normalization constants $\bar{Z}_\phi = \prod_\ell \sqrt{\frac{\pi}{\omega_\ell}}, \bar{Z}_\pi = \prod_\ell \sqrt{\pi\omega_\ell}$

and inverse covariance matrices $\bar{\mathcal{M}}^{-1}_{\ell m} = \frac{2\pi}{\Delta k} 2\omega_\ell \delta_{\ell m}$, $\bar{\mathcal{N}}^{-1}_{\ell m} = \frac{2\pi}{\Delta k}\frac{2}{\omega_\ell}\delta_{\ell m}$

- A coherent state $\rho = |\alpha\rangle\langle\alpha|$ can be obtained by displacing the vacuum, i.e.

$$F_\alpha[\phi] = \bar{Z}_\phi^{\;-1} \exp\left[-\frac{1}{2}\sum_\ell \frac{\Delta k}{2\pi}\sum_m \frac{\Delta k}{2\pi}\, (\phi_\ell - \phi_\ell^\alpha)\, \bar{\mathcal{M}}^{-1}_{\ell m}\, (\phi_m - \phi_m^\alpha)\right],$$

with $\phi_\ell^\alpha = \langle\phi_\ell\rangle_\alpha$, $\pi_\ell^\alpha = \langle\pi_\ell\rangle_\alpha$ and $\alpha_\ell = \frac{1}{\sqrt{2}}(\phi_\ell^\alpha + i\pi_\ell^\alpha)$

Problems in the field theory limit

- For (discretized) theory with $N \in \mathbb{N}$ modes the BBM relation reads Hertz and Cerf '19

$$S[F] + S[G] \geq N(1 + \ln \pi),$$

where we introduced the functional entropy as

$$S[F] = -\int \mathcal{D}\phi \; F[\phi] \, \ln F[\phi]$$

- 1st observation: Right hand side scales with number of modes N

  → Continuum limit and infinite volume limit, which both require $N \to \infty$, lead to divergent bound

- 2nd observation: Independent of the state under consideration, the functional entropy diverges in the field theory limit. For example, for the vacuum we obtain

$$S[\bar{F}] = \ln \bar{Z}_\phi + \frac{1}{2}\int dp \;\, \delta(0) \;\to\; \infty$$

Functional relative entropy

- Similar to divergence of vacuum energy expectation value

$$\bar{E} = \mathrm{Tr}\{\bar{\rho}H\} = \frac{1}{2}\int dp\, \omega(p)\, \delta(0) \;\to\; \infty$$

$\longrightarrow$ A physically reasonable notion of energy has to be formulated as a difference with respect to the vacuum energy

- Can we define entropic uncertainty with respect to some reference state?
- Define functional relative entropy between $F[\phi]$ and some model distribution $\tilde{F}[\phi]$

$$S[F\|\tilde{F}] = \int \mathcal{D}\phi\;\, F[\phi]\, \big(\ln F[\phi] - \ln \tilde{F}[\phi]\big)$$

  - not a true distance measure, but rather a divergence
  - non-negative quantity being zero if and only if the two distributions agree
  - has to be set to $+\infty$ if support condition $\mathrm{supp}(F) \subseteq \mathrm{supp}(\tilde{F})$ is violated

Derivation of the REUR

- Q: What reference model distribution $\tilde{F}[\phi]$ should we choose to describe entropic uncertainty?
  - Q: Which model distributions saturate the entropic uncertainty relation?
  - A: Coherent distributions $F_\alpha[\phi]$, i.e. $S[F_\alpha] + S[G_\alpha] = N(1 + \ln \pi)$
- $F_\alpha[\phi]$ maximizes the functional entropy $S[\tilde{F}]$ for a given covariance matrix $\bar{\mathcal{M}}$ and field expectation value $\phi_\ell^\alpha = \langle \phi_\ell \rangle_\alpha$
- For any distribution $F[\phi]$ with covariance matrix $\mathcal{M}$ and field expectation value $\varphi_\ell = \langle \phi_\ell \rangle$, we have

$$S[F \| F_\alpha] = -S[F] + S[\bar{F}] + \frac{1}{2} \mathrm{Tr}\, \{\bar{\mathcal{M}}^{-1}(\mathcal{M} - \bar{\mathcal{M}})\} + \frac{1}{2} s\, \bar{\mathcal{M}}^{-1}\, s,$$

with $s_\ell = \varphi_\ell - \phi_\ell^\alpha = 0$ for suitably chosen coherent model distribution („optimal coherent model“)

Derivation of the REUR

- Plugging relative entropy $S[F\|F_\alpha] = -S[F] + S[\bar{F}] + \frac{1}{2}\mathrm{Tr}\,\{\bar{\mathcal{M}}^{-1}(\mathcal{M} - \bar{\mathcal{M}})\}$ w.r.t. optimal coherent models into $S[F] - S[\bar{F}] + S[G] - S[\bar{G}] \geq 0$ we obtain our main result, the **relative entropic uncertainty relation (REUR)**

$$S[F\|F_\alpha] + S[G\|G_\alpha] \leq \frac{1}{2}\mathrm{Tr}\,\{\bar{\mathcal{M}}^{-1}(\mathcal{M} - \bar{\mathcal{M}}) + \bar{\mathcal{N}}^{-1}(\mathcal{N} - \bar{\mathcal{N}})\}$$

Arbitrary quantum field theoretic density operator ρ

Optimal coherent models

Upper bound for sum of relative entropies

Bound contains only differences of covariance matrices

$\rightarrow$ No explicit dependence on the number of modes N

$\rightarrow$ Non-trivial statement for entropic uncertainty also for quantum fields

Example: Excitations

- Remember: a free scalar field is just a collection of harmonic oscillators!
- Create excitations / free particles by acting with creation operators on vacuum state

$$\Psi[\phi] = \prod_{k \in \mathfrak{I}} \frac{1}{\sqrt{n_k!}} \left(\sqrt{\frac{\Delta k}{2\pi}} a_k^\dagger \right)^{n_k} \bar{\Psi}[\phi], \quad a_k^\dagger = \frac{1}{\sqrt{2\omega_k}} \left(\omega_k \phi_k - \frac{\delta}{\delta \phi_k} \right)$$

- This yields the functional probability density

$$F[\phi] = |\Psi[\phi]|^2 = \prod_{k \in \mathfrak{I}} \frac{1}{n_k!} H_{n_k}^2 \left(\frac{\phi_k}{\sqrt{\bar{\mathcal{M}}_{kk}}} \right) \bar{F}[\phi]$$

- The covariance matrix of such a state is given by

$$\mathcal{M}_{\ell m} = \int \mathcal{D}\phi \left[\phi_\ell \phi_m \prod_{k \in \mathfrak{I}} \frac{1}{n_k!} H_{n_k}^2 \left(\frac{\phi_k}{\sqrt{\bar{\mathcal{M}}_{kk}}} \right) \right] \bar{F}[\phi] \propto \delta_{\ell m}$$

Example: Excitations

- Using the orthogonality relations as well as recurrence relations of Hermite polynomials, one can show that

for non-excited modes: $\mathcal{M}_{\ell\ell} = \bar{\mathcal{M}}_{\ell\ell}$ for $\ell \notin \mathfrak{I}$

for excited modes: $\mathcal{M}_{\ell\ell} = \bar{\mathcal{M}}_{\ell\ell}\,(1 + 2n_\ell)$ for $\ell \in \mathfrak{I}$

$\rightarrow$ the diagonal components of the vacuum covariance acquire an additive term accounting for the excitations in the excited modes

- Using this result, we can compute the bound of the REUR for the discretized as well as for the continuous theory

$$S[F\|\bar{F}] + S[G\|\bar{G}] \leq \sum_{k\,\in\,\mathfrak{I}} 2n_k$$

$\rightarrow$ This result also holds in the field theory limit!

Summary

- We have presented a relative entropic uncertainty relation (REUR)

$$S[F\|F_\alpha] + S[G\|G_\alpha] \ \leq \ \frac{1}{2}\mathrm{Tr}\ \{\bar{\mathcal{M}}^{-1}(\mathcal{M} - \bar{\mathcal{M}}) + \bar{\mathcal{N}}^{-1}(\mathcal{N} - \bar{\mathcal{N}})\}$$

  describing entropic uncertainty between a scalar field and its conjugate momentum field with respect to optimal coherent states

- The bound of this relation is independent of the number of modes N
- All quantities are well-defined and finite in the field theory limit
- We have demonstrated its properties by considering few particle excitations

Outlook

- Formulate other known entropic uncertainty relations in a field theory sense, e.g., Frank and Lieb relation as well as Wehrl-Lieb inequality (work in progress)
- Extend REUR to include (quantum) memory
- Use REUR to constrain entanglement in quantum field theories

  $\longrightarrow$ obtain criteria being capable of certifying entanglement between spacetime regions?
- Study other field theories: fermions and gauge fields?
- Study interacting theories: perturbation theory and beyond?

Thank you for your attention!

References

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- Walborn *et al.* ´09: Entropic entanglement criteria for continuous variables, Phys. Rev. Lett. 103, 160505 (2009)
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Backup slides

Example: Thermal state

- The thermal density operator is given by $\rho_T = \frac{1}{Z} e^{-\beta H}$, the resulting functional probability density is of Gaussian form

$$F_T[\phi] = \frac{1}{Z_T} \exp\left(-\frac{1}{2} \sum_\ell \frac{\Delta k}{2\pi} \sum_m \frac{\Delta k}{2\pi}\, \phi_\ell \left(\mathcal{M}^T_{\ell m}\right)^{-1} \phi_m\right)$$

with the thermal covariance matrix $\mathcal{M}^T_{\ell m} = (1 + 2 n_{BE}(\omega_\ell))\, \bar{\mathcal{M}}_{\ell m}$

- The bound of the REUR reads

$$\frac{1}{2} \mathrm{Tr}\{\bar{\mathcal{M}}^{-1}(\mathcal{M}^T - \bar{\mathcal{M}})\} = \begin{cases} L \sum_\ell \frac{\Delta k}{2\pi} n_{BE}(\omega_\ell) & \text{continuum limit} \\ 2\pi\delta(0) \int \frac{dp}{2\pi} n_{BE}(\omega(p)) & \text{infinite volume limit} \end{cases}$$

$\rightarrow$ Consider relative entropy densities or finite volume

Example: Thermal state

- As thermal state is of Gaussian form, we can also calculate the LHS of the REUR

$$S[F_T\|\bar{F}] + S[G_T\|\bar{G}] = L\sum_\ell \frac{\Delta k}{2\pi}\,[2n_{BE}(\omega_\ell) - \ln(1 + 2n_{BE}(\omega_\ell))]$$

and plot both sides