Area laws and thermalization from classical entropies

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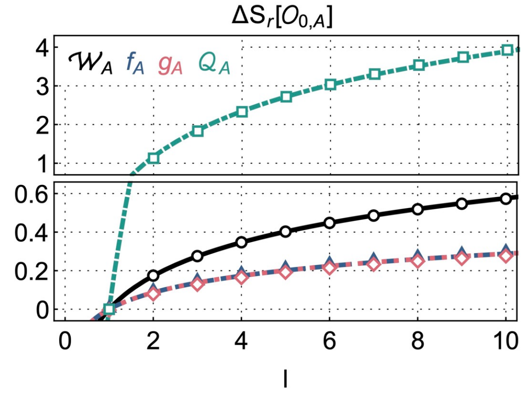

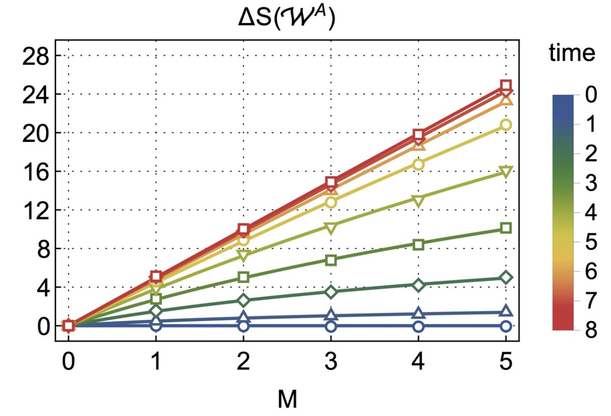

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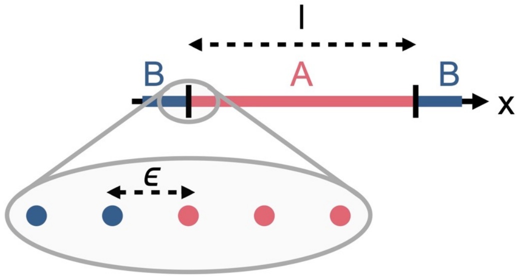
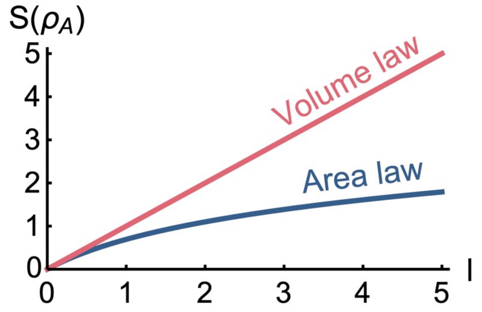
Big picture

- Entanglement of a subregion



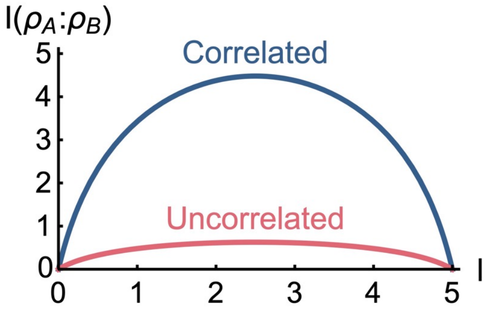
local state $\boldsymbol{\rho}_A = \mathbf{Tr}_B\{\boldsymbol{\rho}\}$



$S(\boldsymbol{\rho}_A) = -\mathbf{Tr}_A\{\boldsymbol{\rho}_A \ln \boldsymbol{\rho}_A\}$

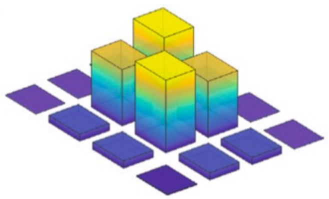
thermal

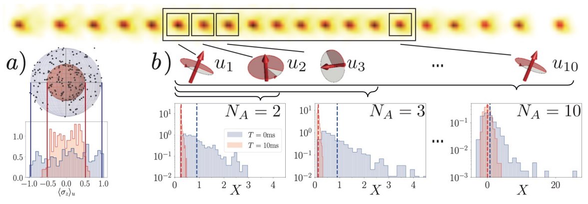
ground, quenched states, …

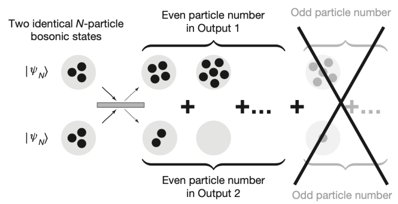


$I(\boldsymbol{\rho}_A\!:\boldsymbol{\rho}_B) = S(\boldsymbol{\rho}_A) + S(\boldsymbol{\rho}_B) - S(\boldsymbol{\rho})$

- Quantum state tomography *or* clever readout







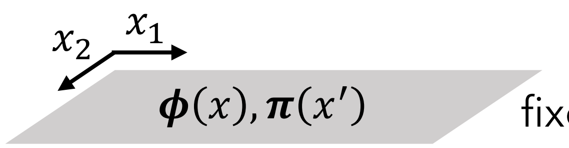
→ Quantum entropies $S(\boldsymbol{\rho}_A)$ and local state $\boldsymbol{\rho}_A$ needed?

Outline

Theory	Experimental perspective
• Subtracted classical entropies	• Bose-Einstein condensate
• Scalar field: Typical states	• Area law → Volume law

Schrödinger picture

- Cauchy hypersurface



fixed t

- Canonical commutation relations

$$[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = i\delta(x - x')$$

- Eigenvalue equations

$$\boldsymbol{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle, \quad \boldsymbol{\pi}(x)|\pi\rangle = \pi(x)|\pi\rangle$$

classical field configurations

eigenstates

Phase-space distributions

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

- Euclidean phase space $\chi = (\phi, \pi)$
- Wigner-W

$$\mathcal{W}[\chi] = \int \frac{\mathcal{D}\phi'}{\pi} \langle \phi - \phi' | \boldsymbol{\rho} | \phi + \phi' \rangle e^{2i \int \mathrm{d}x \, \pi(x) \phi'(x)}$$

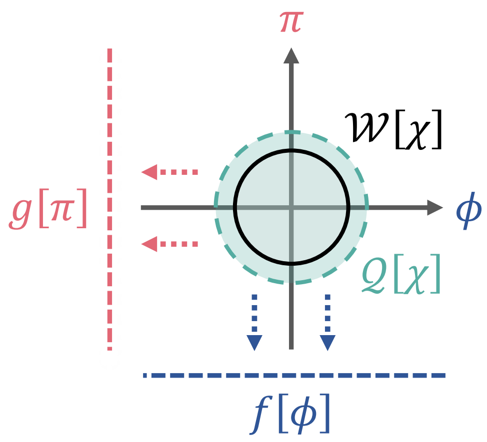
- Marginals

$$f[\phi] = \langle \phi | \boldsymbol{\rho} | \phi \rangle = \int \mathcal{D}\pi \, \mathcal{W}[\chi]$$

$$g[\pi] = \langle \pi | \boldsymbol{\rho} | \pi \rangle = \int \mathcal{D}\phi \, \mathcal{W}[\chi]$$

- Husimi-Q

$$\mathcal{Q}[\chi] = \int \mathcal{D}\chi' \, \mathcal{W}[\chi'] \, \bar{\mathcal{W}}(\chi - \chi')$$



		positive	measurable
theory	W	✗ ?	✗
experiment	f, g	✓	✓
	$\mathcal{Q}$	✓	✓

Classical entropy & uncertainty

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

- Local distributions

$$\mathcal{O}_A[\nu_A] = \int \mathcal{D}\nu_B\, \mathcal{O}[\nu]$$

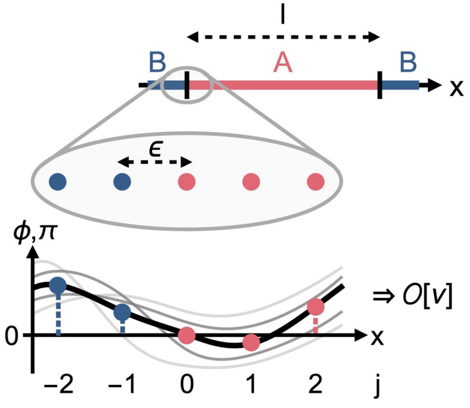
- Classical Rényi entropy

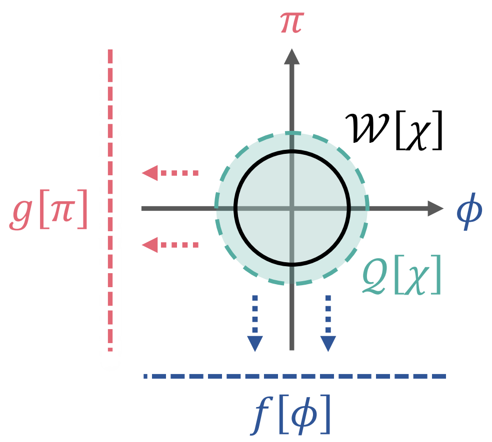
$$S_r[\mathcal{O}_A] = \frac{1}{1-r} \ln\left[\int \mathcal{D}\nu_A\, \mathcal{O}_A^r\right]$$

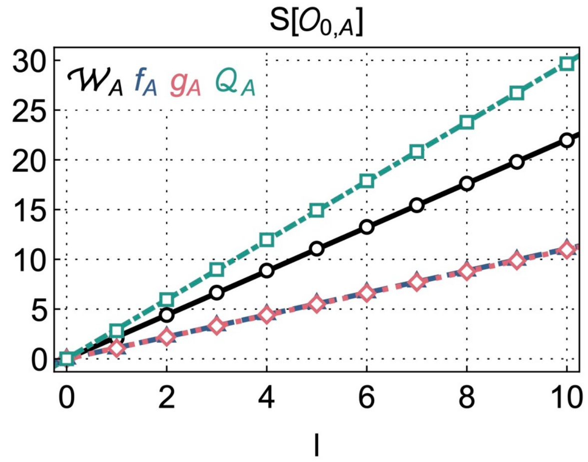
- Entropic uncertainty relations

$$S_r[\mathcal{O}_A] \geq S_r[\bar{\mathcal{O}}_A] \sim \frac{l}{\epsilon}$$

→ Volume law to leading order







Subtracted classical entropies

- Subtract extensive contribution ~ local uncertainty deficit

$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A]$$

- Classical Rényi mutual information ~ classical correlations $A \leftrightarrow B$

$$I_r[\mathcal{O}_A : \mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}]$$
$$= \Delta S_r[\mathcal{O}_A] + \Delta S_r[\mathcal{O}_B] - \Delta S_r[\mathcal{O}]$$

→ Let's check these out!

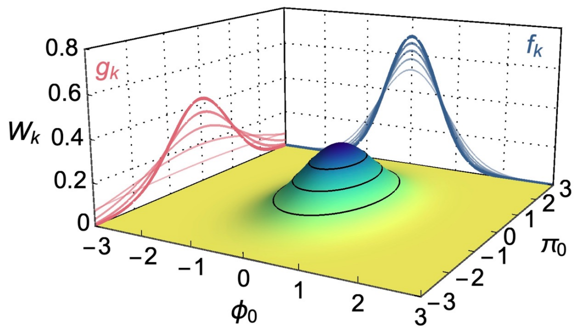
Scalar quantum field

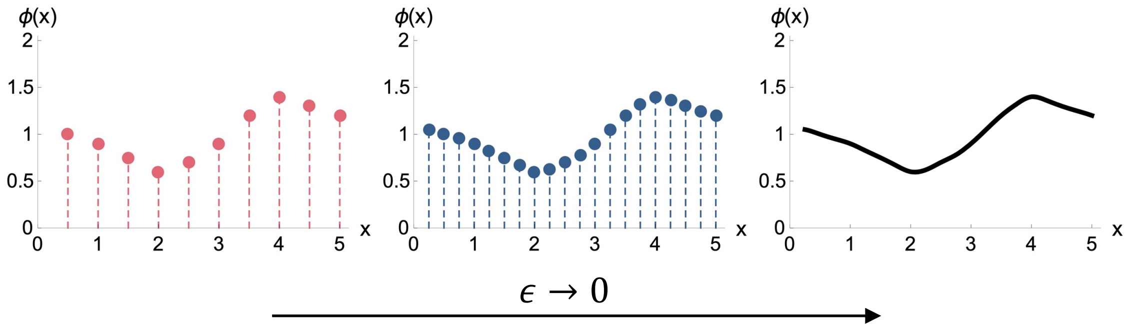
- Hamiltonian

$$\boldsymbol{H} = \int \mathrm{d}x\, [\boldsymbol{\pi}^2 + (\partial_x \boldsymbol{\phi})^2 + m^2 \boldsymbol{\phi}^2]$$

- Typical local distribution

$$\mathcal{O}_A[\nu_A] = \underbrace{\frac{1}{Z_A^{\mathcal{O}}}\, e^{-\frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\, \nu_A^T(x) \left(\gamma_A^{\mathcal{O}}\right)^{-1}(x,x')\, \nu_A(x')}}_{\text{Gaussian}} \times \underbrace{\kappa_A^{\mathcal{O}}[\nu_A]}_{\text{polynomial}}$$



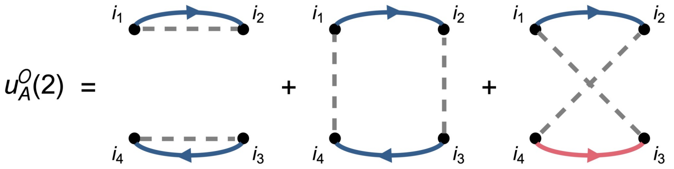


Calculations

Subtracted classical entropies | **Scalar field** | BEC | Area law → Volume law

- Rényi powers

$$\int \mathcal{D}\nu_A\, \mathcal{O}_A^r = \underbrace{\sqrt{\frac{\det^{1-r}\left(2\pi\gamma_A^{\mathcal{O}}\right)}{r^{(2)l/\epsilon}}}}_{\text{Gaussian}} \underbrace{\times\left(\kappa_A^{\mathcal{O}}[\partial_\zeta]\right)^r e^{\frac{1}{2r}\int \mathrm{d}x\mathrm{d}x'\, \zeta^T(x)\gamma_A^{\mathcal{O}}(x,x')\zeta(x')}\Big|_{\zeta=0}}_{\equiv\, U_A^{\mathcal{O}}(r) \text{ non-Gaussian}}$$



- Classical Rényi entropy

$$S_r[\mathcal{O}_A] = \underbrace{\frac{1}{2}\ln\det\left(2\pi\gamma_A^{\mathcal{O}}\right) + \frac{1(2)}{2}\frac{\ln r}{r-1}\frac{l}{\epsilon}}_{\text{Gaussian}} + \underbrace{\frac{\ln U_A^{\mathcal{O}}}{1-r}}_{\text{non-Gaussian}}$$

General formulae

- Subtracted classical Rényi entropy

$$\Delta S_r[\mathcal{O}_A] = \underbrace{\frac{1}{2}\ln\det\left[\gamma_A^{\mathcal{O}}\left(\bar{\gamma}_A^{\mathcal{O}}\right)^{-1}\right]}_{\text{Gaussian}} + \underbrace{\delta S_r[\mathcal{O}_A]}_{\text{non-Gaussian}}$$

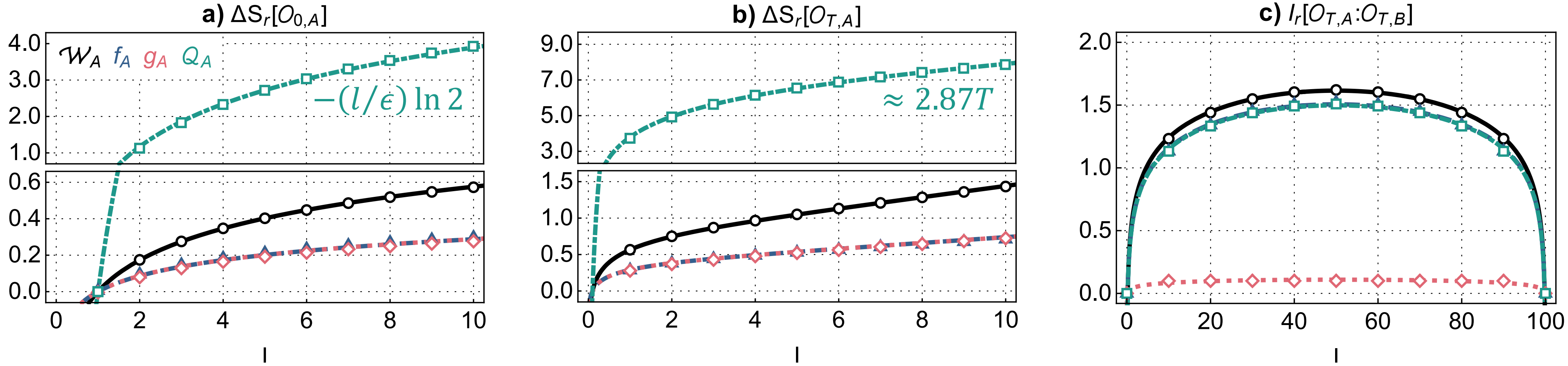
- Classical Rényi mutual information

$$I_r[\mathcal{O}_A\!:\mathcal{O}_B] = \underbrace{\frac{1}{2}\ln\frac{\det(\gamma_A^{\mathcal{O}})\det(\gamma_B^{\mathcal{O}})}{\det(\gamma^{\mathcal{O}})}}_{\text{Gaussian}} + \underbrace{\delta I_r[\mathcal{O}_A\!:\mathcal{O}_B]}_{\text{non-Gaussian}}$$

Gaussian states

- Classical and quantum entropies are related

$$\left.\begin{aligned} S_2(\boldsymbol{\rho}_A) = \Delta S_r[\mathcal{W}_A] &= \Delta S_r[f_A] + \Delta S_r[g_A] \\ I_2[\boldsymbol{\rho}_A : \boldsymbol{\rho}_B] = I_r[\mathcal{W}_A : \mathcal{W}_B] &= I_r[f_A : f_B] + I_r[f_A : g_B] \end{aligned}\right\} \text{ if } \mathcal{W} = f \times g$$



$$\Delta S_r[\mathcal{W}_A] = \frac{c}{4} \ln \frac{l}{\epsilon}$$

$$\Delta S_r[\mathcal{W}_A] = \frac{c}{4} \ln \frac{\sinh(\pi T l)}{\pi \epsilon T}$$

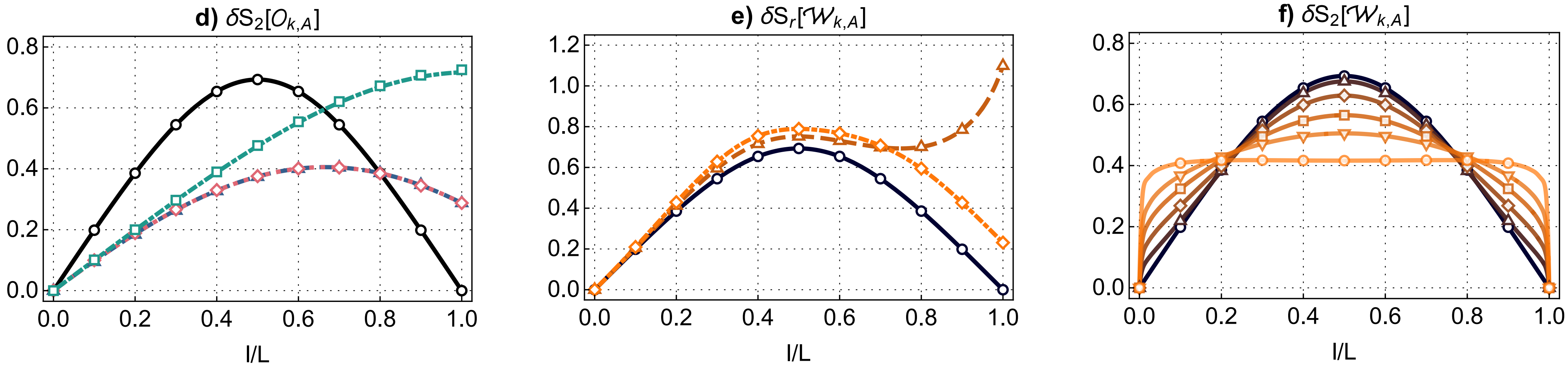
$$I_r\left[\mathcal{O}_{T,A} : \mathcal{O}_{T,B}\right] \leq a\, |\partial A|$$

Particles

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

$$\delta S_r[\mathcal{O}_{k,A}] = \frac{1}{1-r} \ln \left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i \right]$$



$$\delta S_2[\mathcal{W}_{k,A}] = -\ln \left[\left(\frac{l}{L}\right)^2 + \left(1 - \frac{l}{L}\right)^2 \right]$$

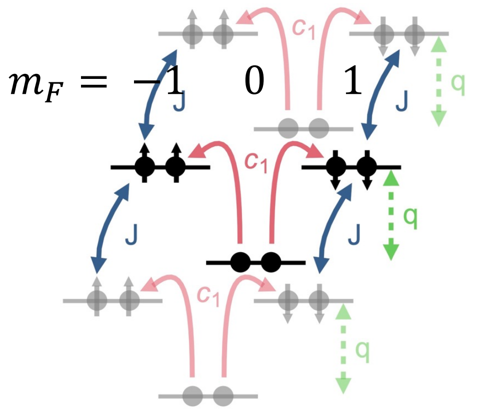
$$\delta S_r[\mathcal{O}_{k,A}] \sim (2)\frac{l}{L}$$
$$\sim \delta S_2[\boldsymbol{\rho}_{k,A}]$$

small energies
$$\delta S_2[\mathcal{W}_{k,A}] \to const.$$

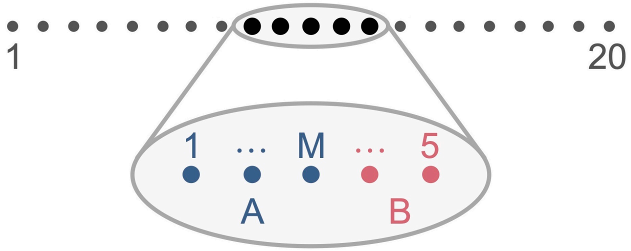
Setup

- Multi-well BEC with ^{7}Li atoms ($F = 1, m_F = -1,0,1$)
- Hamiltonian

$$\boldsymbol{H} = \sum_{j=1}^{20} q\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right) + c_0 \boldsymbol{N}^j\left(\boldsymbol{N}^j - \mathbb{I}\right)$$
$$+c_1\left[\left(\boldsymbol{N}_0^j - \tfrac{1}{2}\mathbb{I}\right)\left(\boldsymbol{N}_1^j + \boldsymbol{N}_{-1}^j\right) + \boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_0^{j\dagger}\boldsymbol{a}_1^j\boldsymbol{a}_{-1}^j + h.c.\right]$$
$$-J\sum_{j=1}^{19}\sum_{m_F=\pm 1}\left(\boldsymbol{a}_{m_F}^{j\dagger}\boldsymbol{a}_{m_F}^{j+1} + h.c.\right)$$



- Local subregion after quench (q, J) of pure vacuum



Routine

- Spin observables

$$\boldsymbol{\phi}^j \equiv \frac{1}{\sqrt{2n}} \boldsymbol{S}_x^j = \frac{1}{\sqrt{2n}} \frac{1}{\sqrt{2}} \left[\boldsymbol{a}_0^{j\dagger}(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j) + h.c.\right]$$

$$\boldsymbol{\pi}^j \equiv \frac{-1}{\sqrt{2n}} \boldsymbol{Q}_{yz}^j = \frac{1}{\sqrt{2n}} \frac{-i}{\sqrt{2}} \left[\boldsymbol{a}_0^{j\dagger}(\boldsymbol{a}_1^j + \boldsymbol{a}_{-1}^j) - h.c.\right]$$

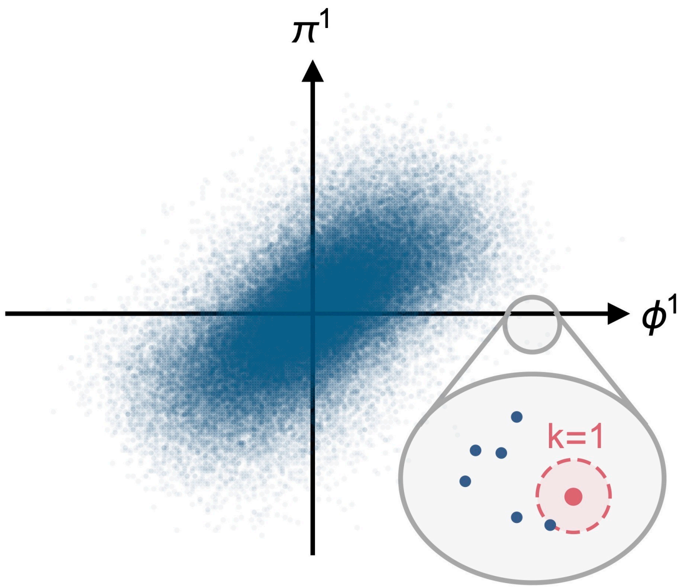
- k-nearest neighbor (kNN) method

$$\hat{S}(k, N_s) = g(k, N_s, d) + \frac{d}{N_s} \sum_{i=1}^{N_s} \ln \epsilon^i(k)$$

→ Asymptotically unbiased

→ Entropy estimated *directly* from data

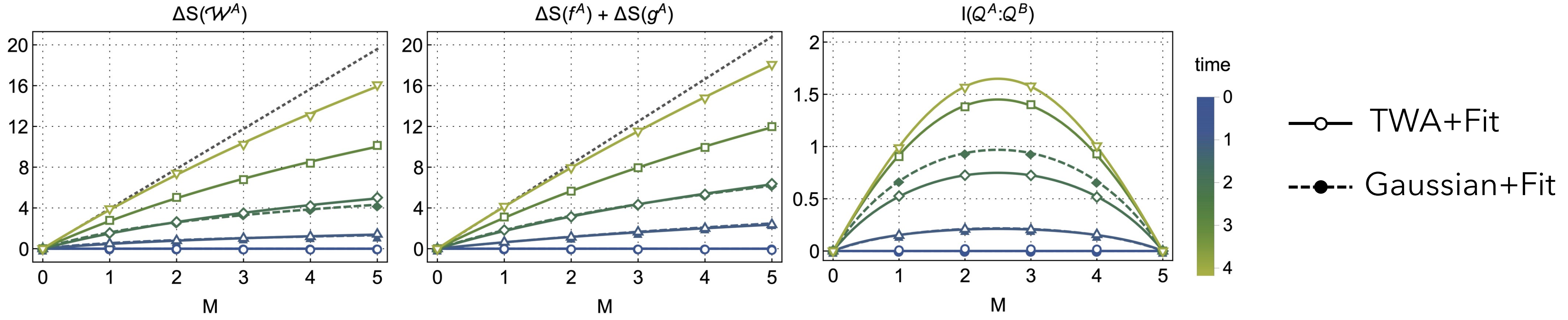
→ *No* assumptions on $\boldsymbol{\rho}_A$ or $\mathcal{O}_A$



Area law

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law

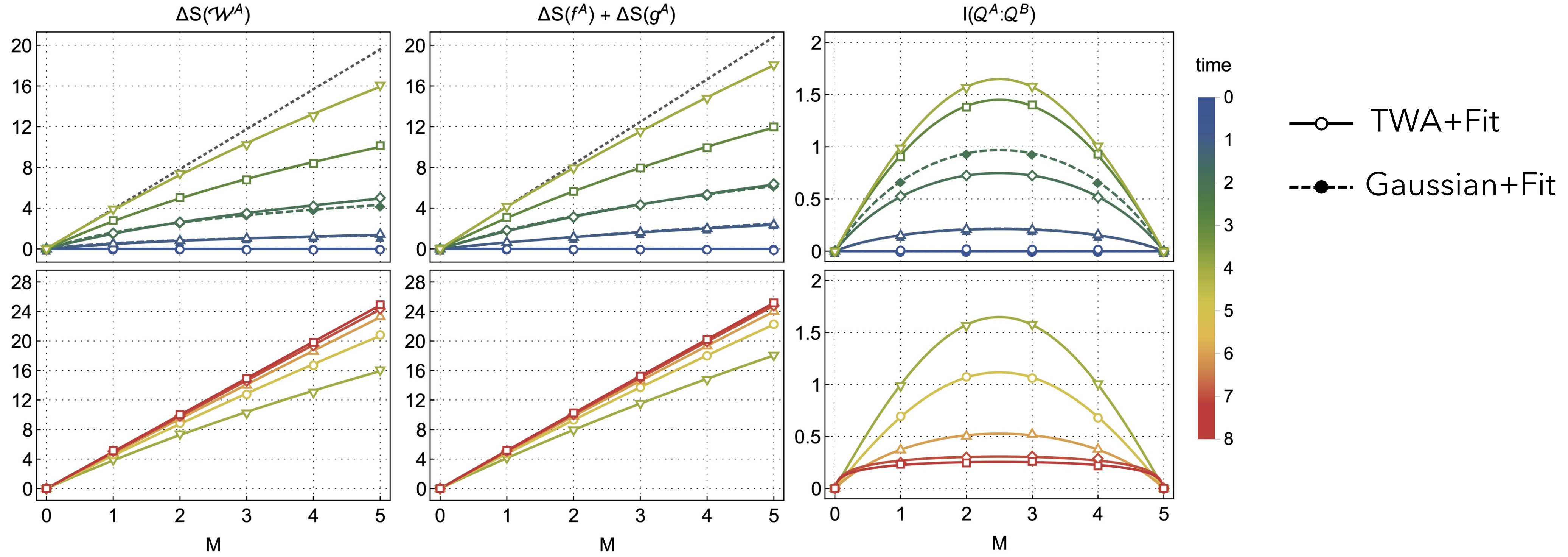
- Early times



→ Area laws signal build-up of quantum correlations

Area law → Volume law

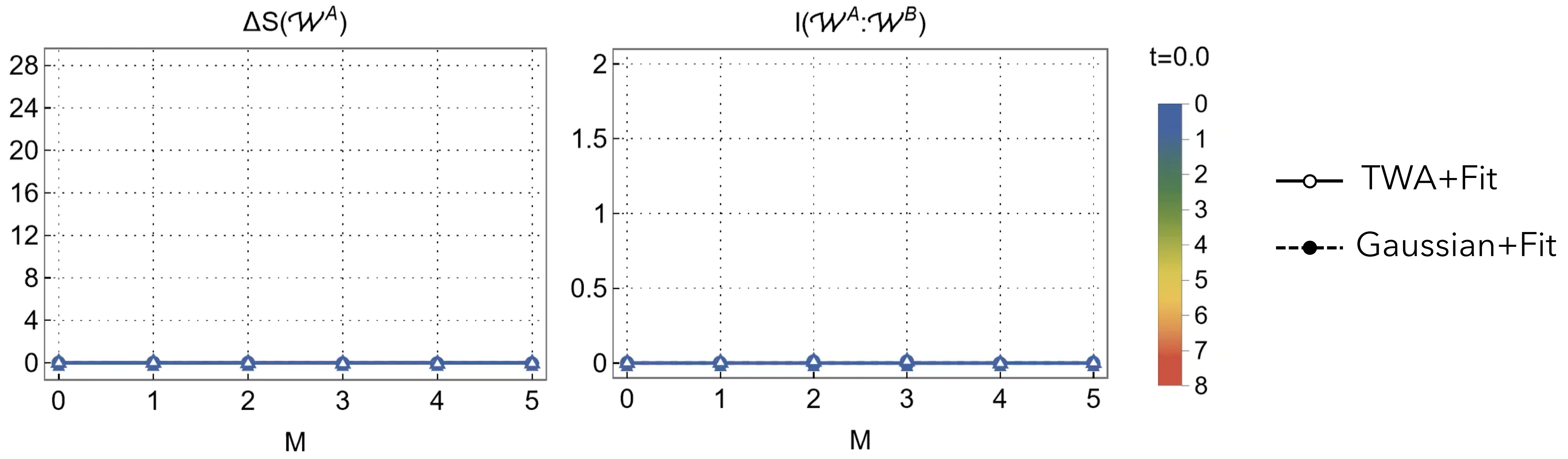
- Early times + late times



→ Volume laws signal local thermalization, incline = $1/T$

Area law → Volume law

- Full time evolution of $\mathcal{W}$-quantities



Summary

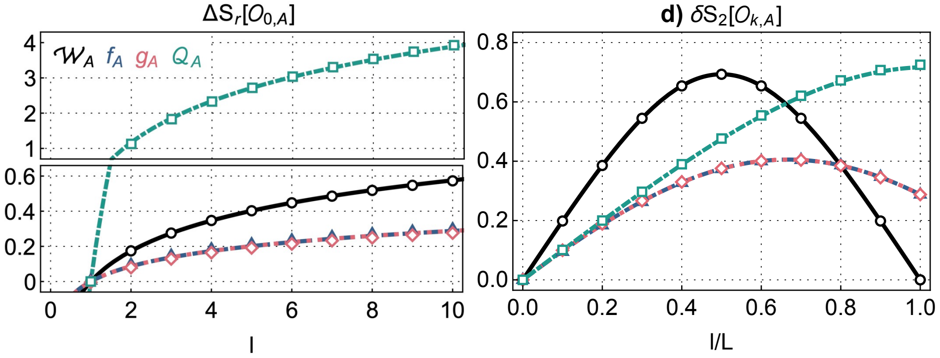
Theory

- Subtracted classical entropies

$$\Delta S_r[\mathcal{O}_A] = S_r[\mathcal{O}_A] - S_r[\bar{\mathcal{O}}_A]$$

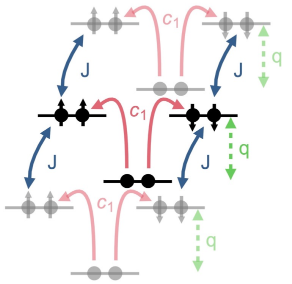
$$I_r[\mathcal{O}_A\!:\!\mathcal{O}_B] = S_r[\mathcal{O}_A] + S_r[\mathcal{O}_B] - S_r[\mathcal{O}]$$

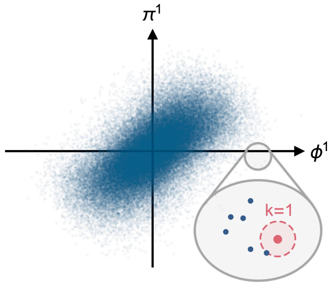
- Scalar field: Typical states



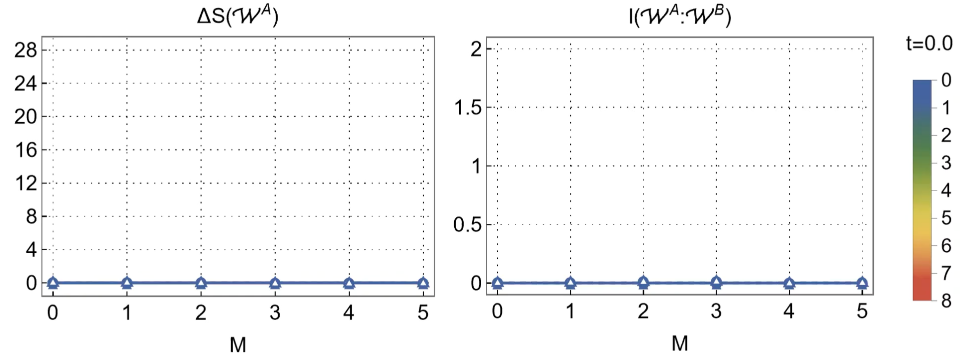
Experimental perspective

- Bose-Einstein condensate



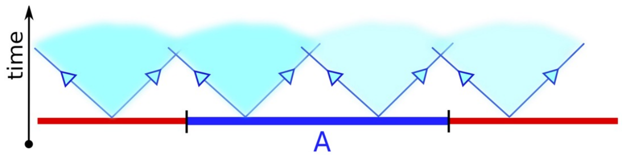


- Area law → Volume law

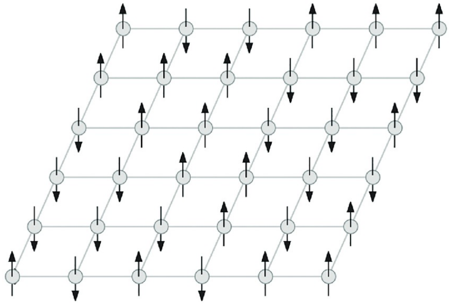


Outlook

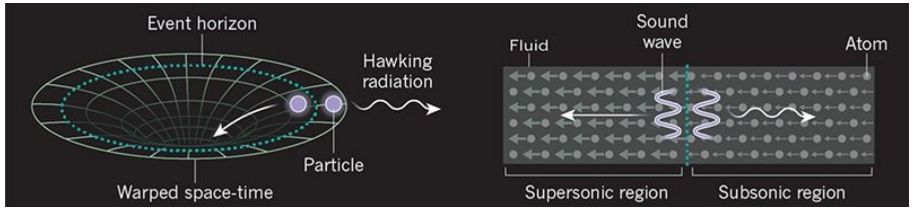
- Classical entropies solve *most* of their quantum analog's problems



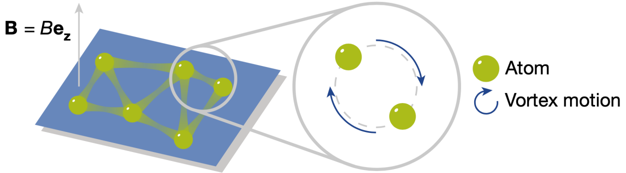
Information spreading/scrambling,
Lieb-Robinson bound for MI



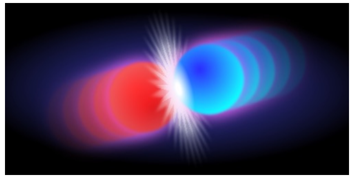
Quantum phase transitions
Many-body localization



Entropy of sonic black holes



Topological EE



Interacting theories

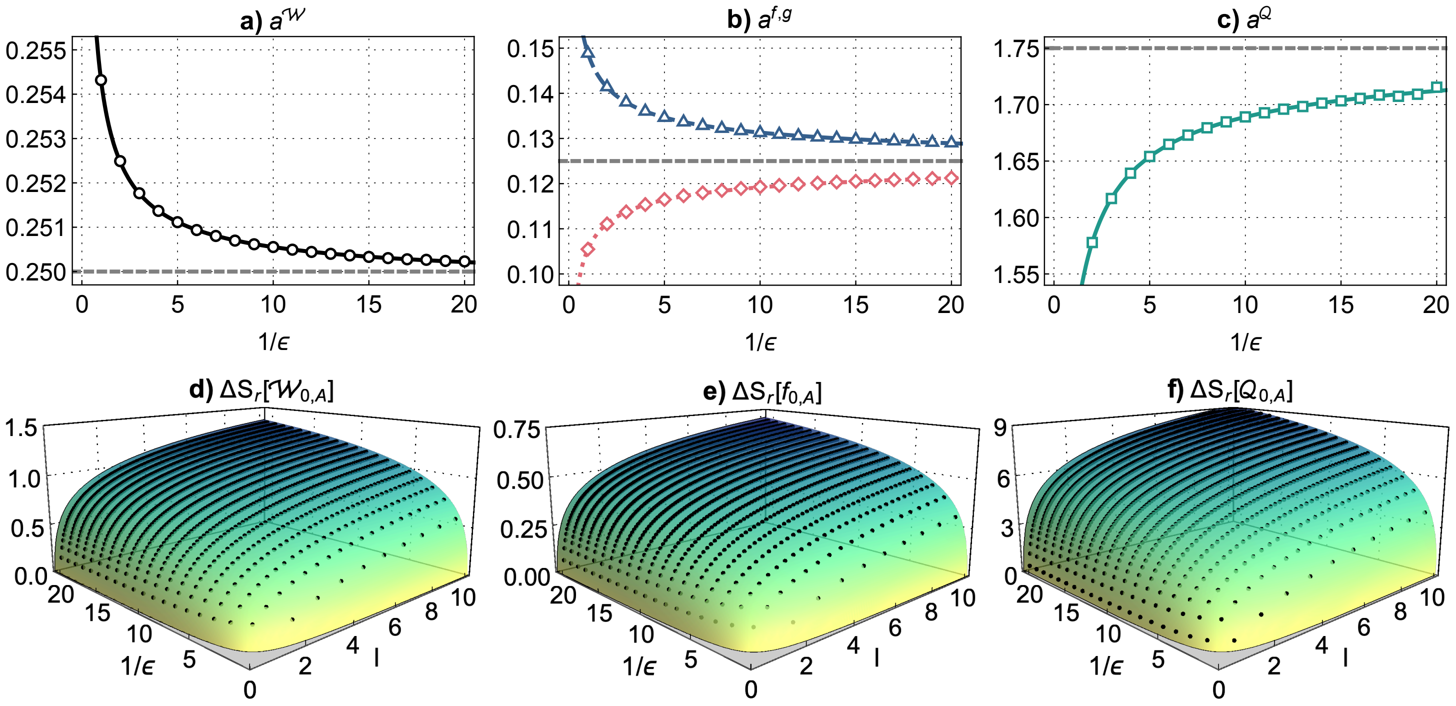
Thanks
for your
attention :)

Backup

Central charge

- Ground state entropies

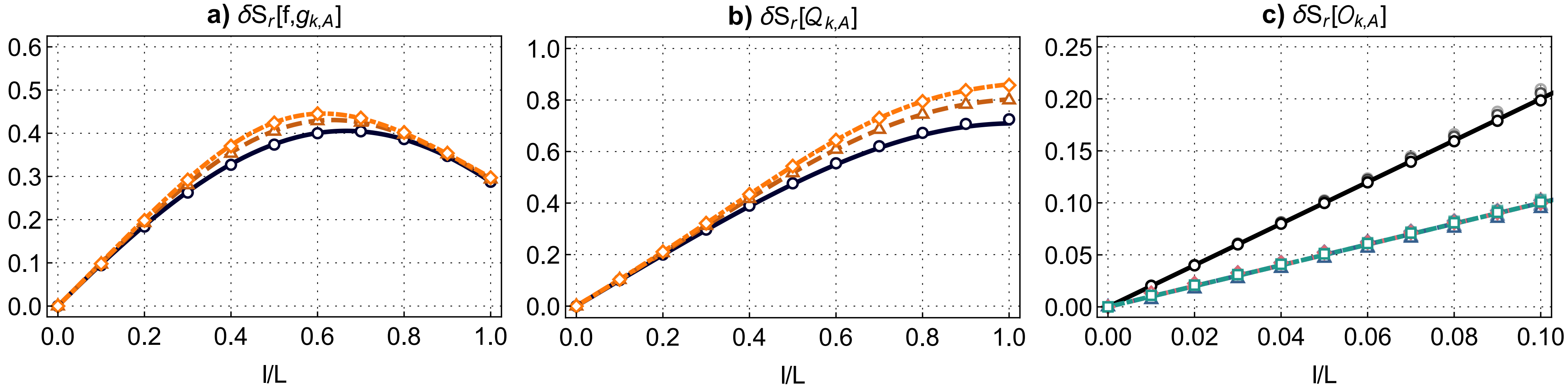
$$\Delta S_r[\mathcal{W}_{0,A}] = \Delta S_r[f_{0,A}] + \Delta S_r[g_{0,A}] = \frac{c}{4}\ln\frac{l}{\epsilon}$$



Particles – Other quantities

- High particle energies: $\omega(p) = \sqrt{m^2 + p^2} \gg \frac{1}{l}, \frac{1}{L-l}$

$$\delta S_r[\mathcal{O}_{k,A}] = \frac{1}{1-r} \ln\left[1 + \sum_{i=1}^{r} a_{r,i} \left(\frac{l}{L}\right)^i\right]$$

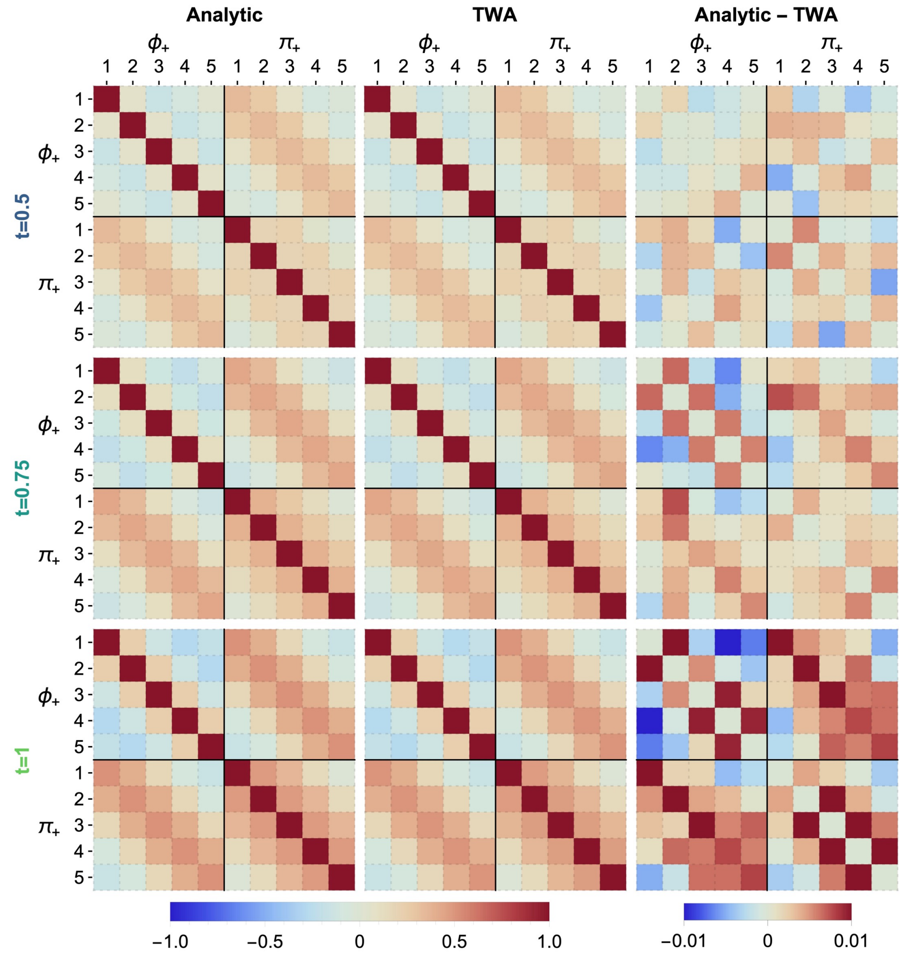


Parameters

- Experimental values
  - Total system size $N = 20$
  - Subsystem size $M = 5$
  - Energy scale $c_1 = -1/n$
  - Number of atoms $n = 10^3$ per well
  - Spin coupling $c_0 = -2c_1$ (^{7}Li)
  - 2nd order Zeemann shift $q = 2J$
  - Coupling between wells $J = 2$
  - Samples $N_s = 10^4$

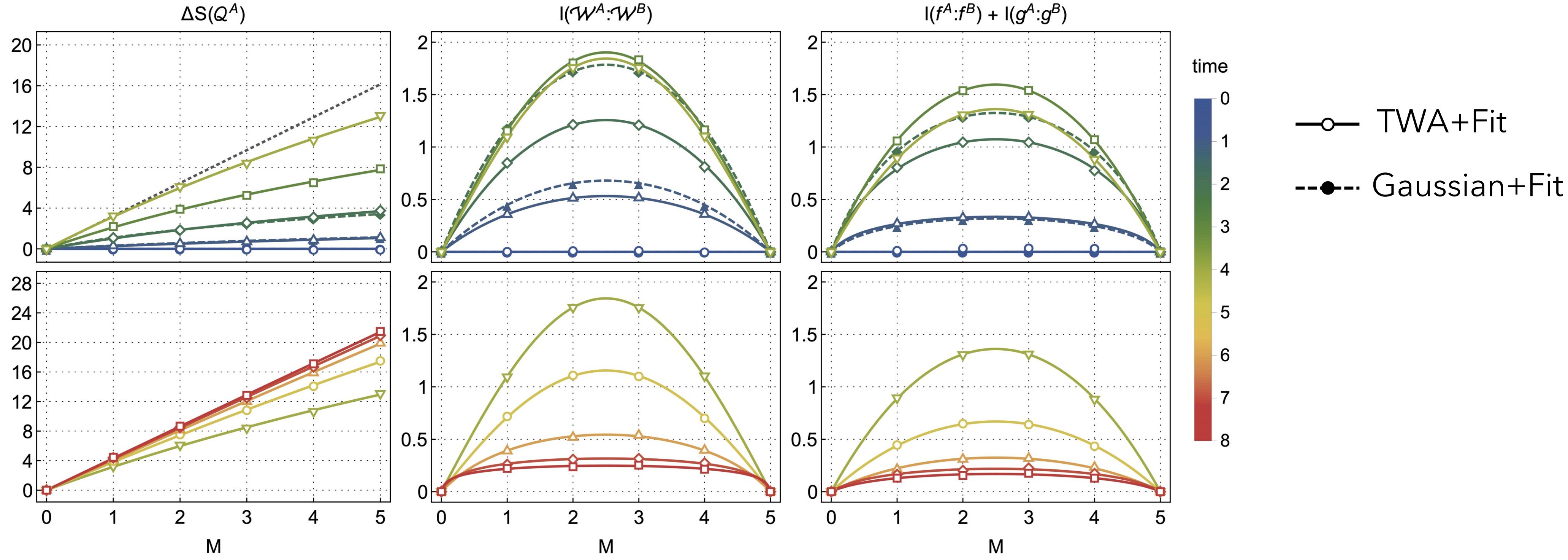
Correlation matrices: Gaussian vs. TWA

Subtracted classical entropies | Scalar field | BEC | Area law → Volume law | Backup



Time evolution - Other quantities

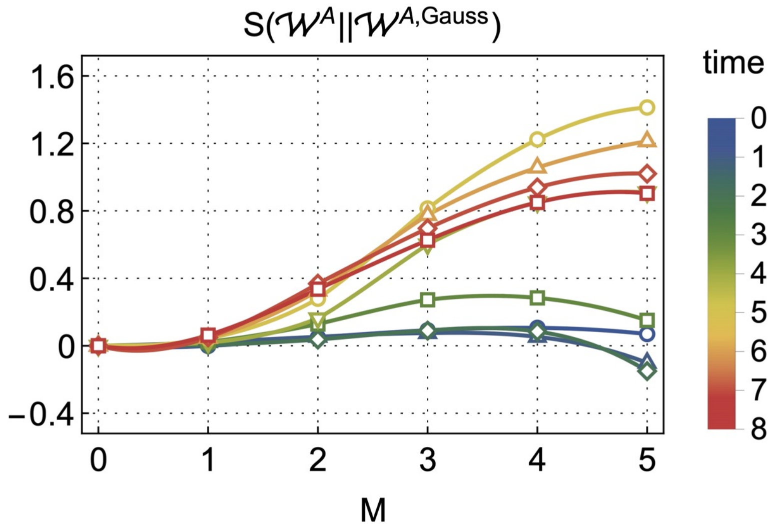
- Early times + late times



Non-Gaussianity

- Relative Wigner entropy

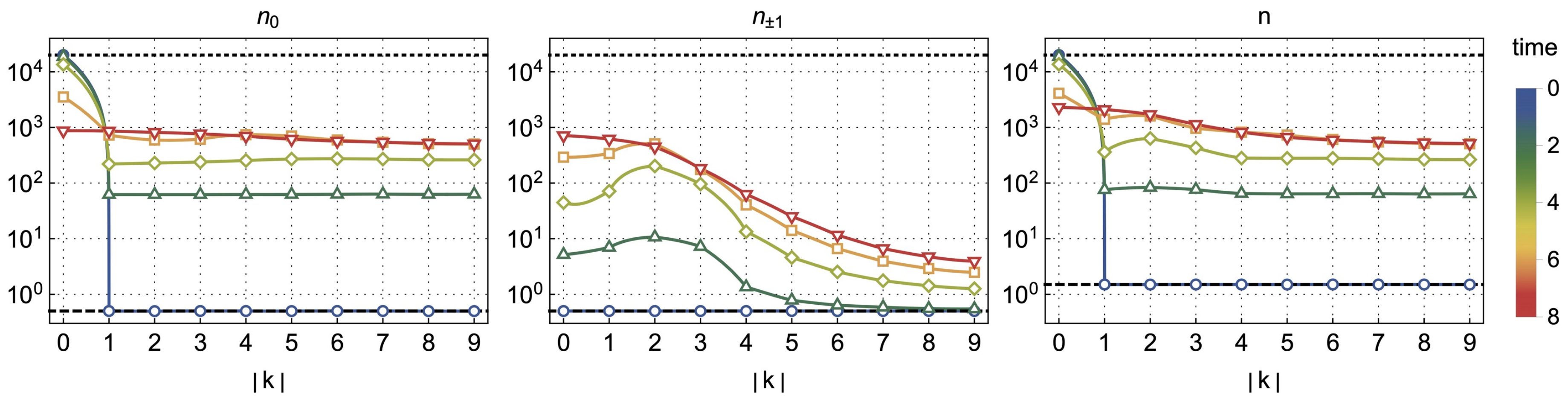
$$S\left[\mathcal{W}^A \| \mathcal{W}^{A,\mathrm{Gauss}}\right] = \int \mathcal{D}\nu_A \, \mathcal{W}^A \left(\ln \mathcal{W}^A - \ln \mathcal{W}^{A,\mathrm{Gauss}}\right)$$



→ Non-Gaussian features in higher dimensions

Mode occpuations

- Populations of momentum modes



→ Mesoscopic occupations justify TWA for late times