Information and majorization theory in fermionic phase space

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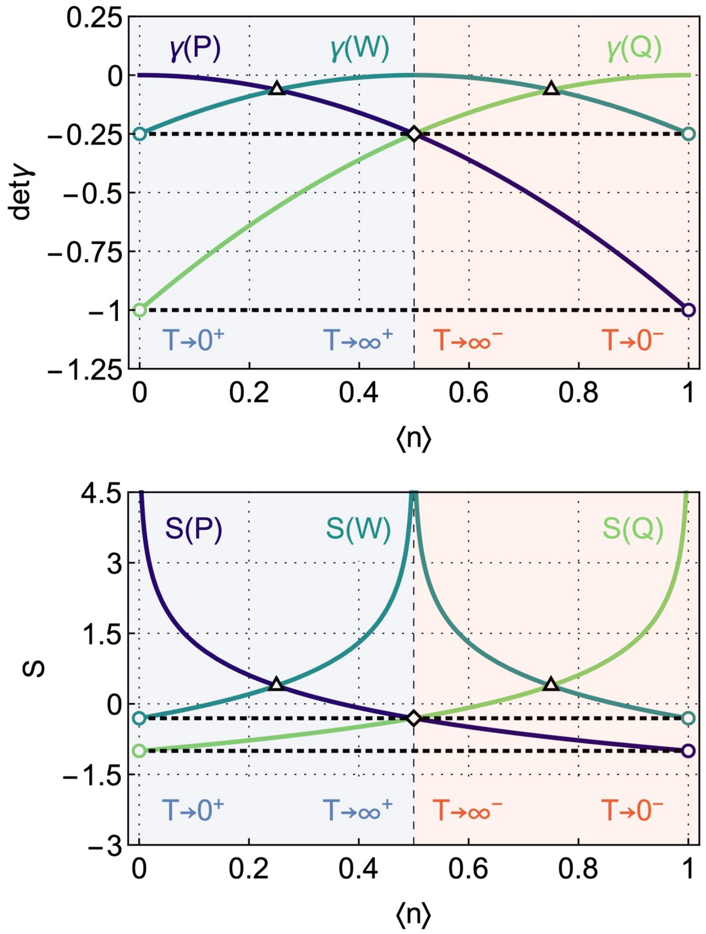
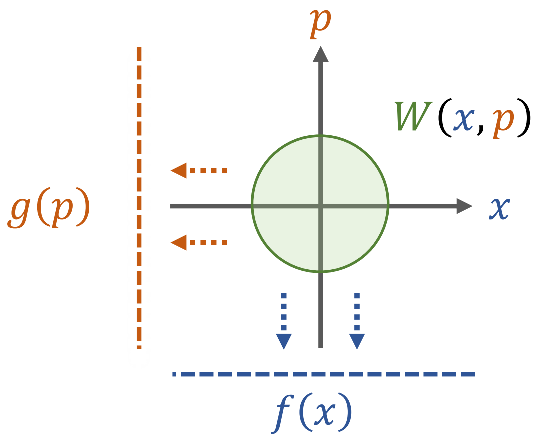


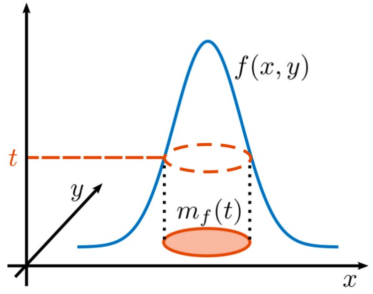
Big picture

- Bosons: Phase-space methods, uncertainty relations, majorization, …



$$\sigma_x \sigma_p \geq \frac{1}{2} \qquad \det \gamma = \begin{vmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{vmatrix} \geq \frac{1}{4}$$

$$S(f) + S(g) \geq 1 + \ln \pi \qquad S(W) \geq 1 + \ln \pi \;? \qquad S(Q) \geq 1$$



$$W^+ \prec W_0 \;?$$

- What about *fermions*?

Single fermionic mode

- Pauli's principle
  - Anti-commutation relations $\{\boldsymbol{a}, \boldsymbol{a}^\dagger\} = \mathbb{I}, \; \{\boldsymbol{a}, \boldsymbol{a}\} = \{\boldsymbol{a}^\dagger, \boldsymbol{a}^\dagger\} = 0$
  - Two-dimensional Hilbert space $\mathcal{H}_2 = \{|0\rangle, |1\rangle\}$
  - Finite mean particle numbers $\langle n \rangle = \mathbf{Tr}\{\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\} \in [0,1]$
- Most general state Cahill '99

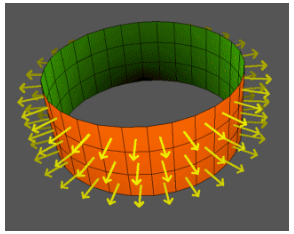
$$\boldsymbol{\rho} = (1 - \langle n \rangle)\boldsymbol{a}\boldsymbol{a}^\dagger + \lambda\boldsymbol{a} + \lambda^*\boldsymbol{a}^\dagger + \langle n \rangle\boldsymbol{a}^\dagger\boldsymbol{a}$$

with $\lambda \in \mathbb{C}, |\lambda| \leq \sqrt{\langle n \rangle(1 - \langle n \rangle)}$ s.t. $\boldsymbol{\rho} \geq 0$

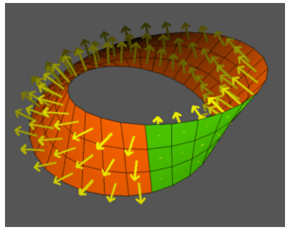
Physical states and Gaussianity

- Physical states should be invariant under 2π rotation (up to phase)[Friis '16]

$|0\rangle \rightarrow |0\rangle$



$|1\rangle \rightarrow -|1\rangle$

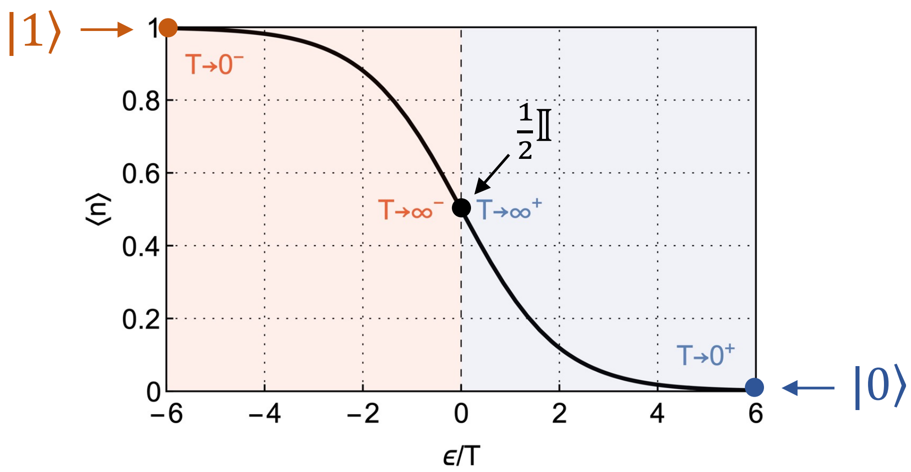


- Physical states = no superpositions = thermal Gaussian states

$$\boldsymbol{\rho} = (1 - \langle n \rangle)\boldsymbol{a}^\dagger \boldsymbol{a} + \langle n \rangle \boldsymbol{a}^\dagger \boldsymbol{a}$$

$$= \frac{1}{1+e^{\nu}} e^{\nu \boldsymbol{a}^\dagger \boldsymbol{a}}$$

where $\nu = -\frac{\epsilon}{T}$



Fermionic coherent states

- Displacement operator[Cahill '99]

$$\boldsymbol{D}(\alpha) = e^{\boldsymbol{a}^\dagger\alpha - \alpha^*\boldsymbol{a}} = \mathbb{I} + \boldsymbol{a}^\dagger\alpha - \alpha^*\boldsymbol{a} + \left(\tfrac{1}{2}\mathbb{I} - \boldsymbol{a}^\dagger\boldsymbol{a}\right)\alpha\alpha^*$$

with Grassmann-valued displacements α, α^*

$$\{\alpha, \alpha^*\} = \{\alpha, \alpha\} = \{\alpha^*, \alpha^*\} = 0, \qquad \{\alpha, \boldsymbol{a}\} = \{\alpha, \boldsymbol{a}^\dagger\} = \{\alpha^*, \boldsymbol{a}\} = \{\alpha^*, \boldsymbol{a}^\dagger\} = 0$$

- Fermionic coherent state

$$|\alpha\rangle = \boldsymbol{D}(\alpha)|0\rangle$$

- Berezin integrals = differentiation

$$\int \mathcal{D}\alpha\, \alpha\alpha^* \equiv \int \mathrm{d}\alpha^*\mathrm{d}\alpha\, \alpha\alpha^* = +1$$

Phase-space distributions

- Glauber-P

$$\boldsymbol{\rho} = \int \mathcal{D}\alpha\, P(\alpha)|\boldsymbol{\alpha}\rangle\langle-\boldsymbol{\alpha}| \quad \to \quad P(\alpha) = -\langle n\rangle e^{-\frac{\alpha\alpha^*}{\langle n\rangle}} \qquad = -\langle n\rangle + \underline{\alpha\alpha^*}$$

- Wigner-W

$$W(\alpha) = \int \mathcal{D}\beta\, e^{\alpha\beta^* - \beta\alpha^*}\mathbf{Tr}\{\boldsymbol{\rho}\boldsymbol{D}(\alpha)\} = \left(\tfrac{1}{2} - \langle n\rangle\right)e^{\frac{\alpha\alpha^*}{\frac{1}{2}-\langle n\rangle}} \quad = \tfrac{1}{2} - \langle n\rangle + \underline{\alpha\alpha^*}$$

- Husimi-Q

$$Q(\alpha) = \langle\alpha|\boldsymbol{\rho}|-\alpha\rangle \qquad\qquad = (1 - \langle n\rangle)e^{\frac{\alpha\alpha^*}{1-\langle n\rangle}} \quad = 1 - \langle n\rangle + \underline{\alpha\alpha^*}$$

- All normalized

$$\int \mathcal{D}\alpha\, P(\alpha) = \int \mathcal{D}\alpha\, W(\alpha) = \int \mathcal{D}\alpha\, Q(\alpha) = \mathbf{Tr}\{\boldsymbol{\rho}\} = 1$$

Supernumbers

- Elements of the Grassmann algebra = supernumbers[deWitt '92]

$$z = \underbrace{c_0}_{\text{body } z_B} + \underbrace{c_1\alpha + c_2\alpha^* + c_3\alpha\alpha^*}_{\text{soul } z_S} \qquad c_i \in \mathbb{C}$$

- Properties mainly dictated by body

$$z \in \mathbb{R} \;\Leftrightarrow z_B \in \mathbb{R}, \quad z_1 \gtrless z_2 \Leftrightarrow z_{1,B} \gtrless z_{2,B}, \quad \int \mathcal{D}\alpha\, z = c_3$$

- Phase-space distributions

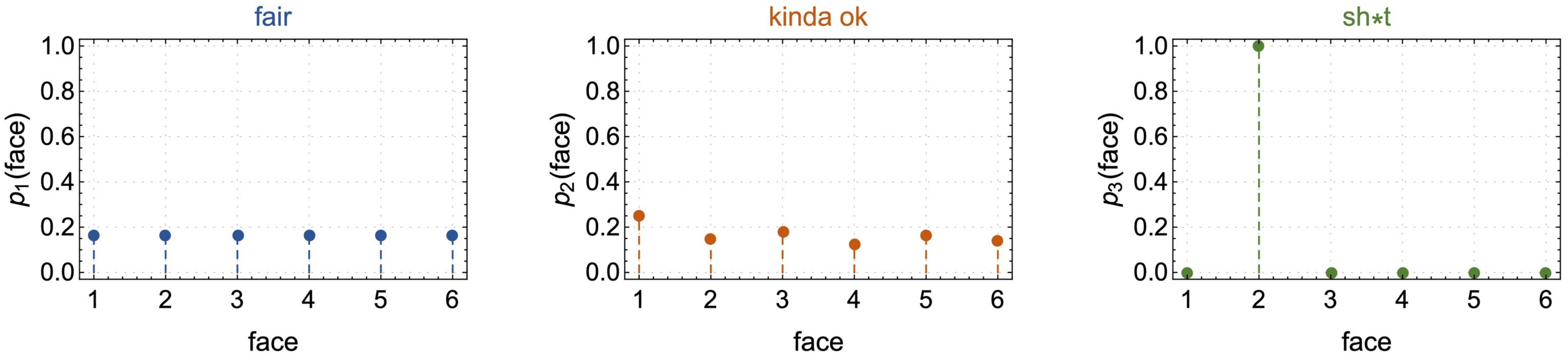
$$P(\alpha) = -\langle n \rangle + \alpha\alpha^*$$
$$W(\alpha) = \tfrac{1}{2} - \langle n \rangle + \alpha\alpha^*$$
$$Q(\alpha) = 1 - \langle n \rangle + \alpha\alpha^*$$

- equal souls, different bodies
- ordered: $Q(\alpha) > W(\alpha) > P(\alpha) \;\; \forall\alpha$
- definite sign: $Q(\alpha) \geq 0,\; P(\alpha) \leq 0,\; W(\alpha) \gtreqless 0 \Leftrightarrow \langle n \rangle \lesseqgtr \frac{1}{2}$

Majorization theory

Physical states | Phase space | **Majorization relations** | Uncertainty relations

- How *ordered* is a probability distribution? Marshall '11



$$p_1 = \left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right) \prec p_2 = (0.25,0.15,0.18,0.12,0.17,0.13) \prec p_3 = (0,1,0,0,0,0)$$

	disorder	→	order
⇒	high	entropy	low
⇒	high	variance	low

Majorization relations

- z_1 is majorized by z_2, written as $z_1 \prec z_2$, if

$$\int \mathcal{D}\alpha\, f(z_1) \geq \int \mathcal{D}\alpha\, f(z_2)$$

  for all *concave* $f\colon \mathcal{I}(z_B) \to \mathbb{R}$ with $f(0) = 0$

- Phase-space majorization relations (subindex $\langle n \rangle$)
  - Glauber-P $\quad P_1 \prec P \prec P_0$
  - Wigner-W $\quad W_1 \prec W^- \prec W_{1/2} \prec W^+ \prec W_0$
  - Husimi-Q $\quad Q_1 \prec Q \prec Q_0$

Bosons

- $\int \mathcal{D}\alpha\, f(P) \notin \mathbb{R}$ ✘
- $W^+ \prec W_0$ Van Herstraeten '23 ?
- $Q \prec Q_0$ Lieb '14 ✓

→ Vacuum $|0\rangle$ is most ordered, excited state $|1\rangle$ most disordered

Proof

- Superfunctional $f(z)$ defined via Taylor series deWitt '92

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(z_B)\, z_S^j$$

- For phase-space distributions: $z_S^j = (\alpha\alpha^*)^j = 0$ when $j > 1$

$$f(z) = f(z_B) + f'(z_B)\alpha\alpha^*$$

Remark: f' exists almost everywhere by Rademacher's theorem for Lipschitz-concave f

→ Concave averages simplify to $\int \mathcal{D}\alpha\, f(z) = f'(z_B)$

- Concavity ⇒ monotonicity of the derivative $z_{1,B} \leq z_{2,B} \Rightarrow f'(z_{1,B}) \geq f'(z_{2,B})$

→ $z_{1,B} \leq z_{2,B} \Rightarrow \int \mathcal{D}\alpha\, f(z_1) \geq \int \mathcal{D}\alpha\, f(z_2)$ ■

Second moments

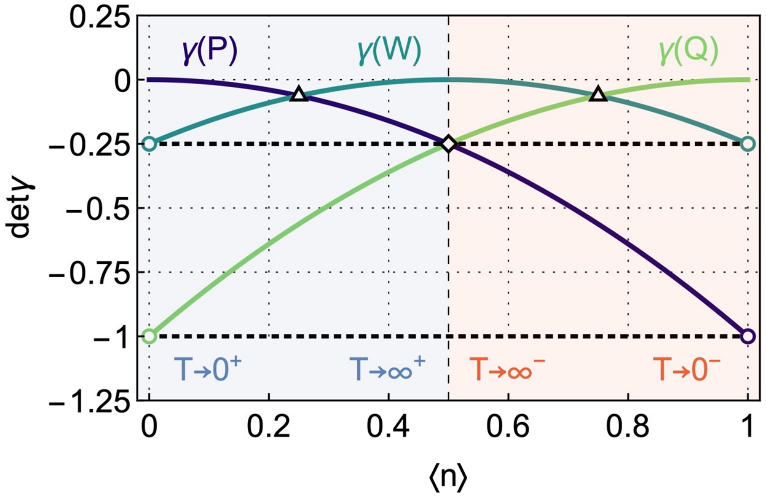
- Covariance matrix

$$\gamma_{jj'}(z) = \frac{1}{2i}\int \mathcal{D}\alpha\, z(\alpha)\big[\alpha_j, \alpha_{j'}\big] = z_B(\sigma_y)_{jj'}$$

- Determinant bounded below by uncertainty principle

$$\det\gamma(z) = -z_B^2 \geq -\max z_B^2$$

- In phase space
  - $\det\gamma(P) = -\langle n\rangle^2 \qquad \geq \det\gamma(P_1) = -1$
  - $\det\gamma(W) = -\left(\frac{1}{2} - \langle n\rangle\right)^2 \geq \det\gamma\left(W_{0,1}\right) = -\frac{1}{4}$
  - $\det\gamma(Q) = -(1-\langle n\rangle)^2 \geq \det\gamma(Q_0) = -1$



Entropies

Physical states | Phase space | Majorization relations | Uncertainty relations

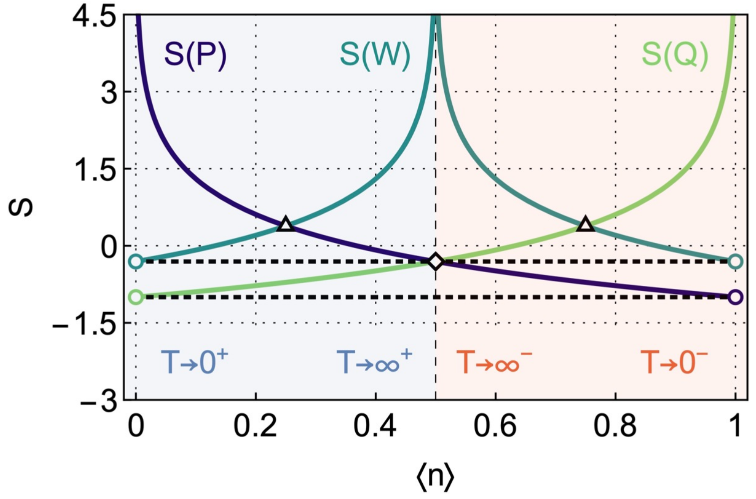
- Rényi entropy of order $r \in (0,1) \cup (1,\infty)$

$$S_r(z) = \frac{1}{1-r} \ln\left[\int \mathcal{D}\alpha \, |z(\alpha)|^r\right] = \frac{\ln r}{1-r} - \ln|z_B|$$

- Entropic uncertainty relations

$$S_r(z) \geq \frac{\ln r}{1-r} - \max \ln|z_B|$$

- In phase space
  - $S_r(P) = \frac{\ln r}{1-r} - \ln|\langle n\rangle| \geq S_r(P_1) = \frac{\ln r}{1-r}$
  - $S_r(W) = \frac{\ln r}{1-r} - \ln\left|\frac{1}{2} - \langle n\rangle\right| \geq S_r\left(W_{0,1}\right) = \frac{\ln r}{1-r} + \ln 2$
  - $S_r(Q) = \frac{\ln r}{1-r} - \ln|1 - \langle n\rangle| \geq S_r(Q_0) = \frac{\ln r}{1-r}$



Equivalences

- All physical states are Gaussian

  → All types of relations are equivalent!

- Comparison
  - Majorization $z_1 \prec z \prec z_0$

    $\Updownarrow$ Variable transformation in Berezin integral

  - Second moments $-\min z_B^2 \geq \det \gamma(z) \geq -\max z_B^2$

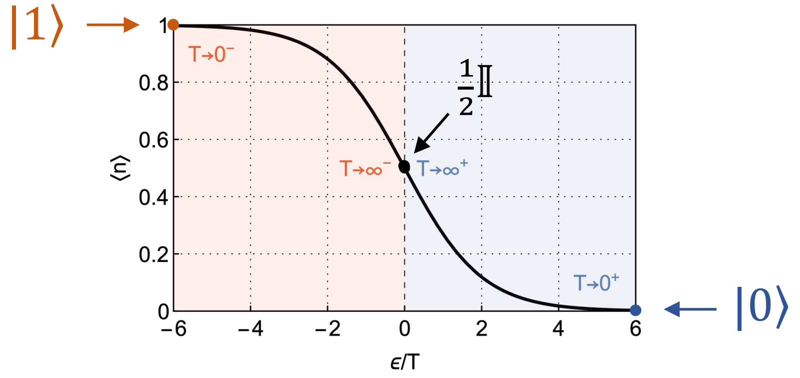
    $\Updownarrow$ $S_r(z) = \frac{\ln r}{1-r} - \frac{1}{2}\ln[-\det \gamma(z)]$

  - Entropies $\frac{\ln r}{1-r} - \min \ln|z_B| \geq S_r(z) \geq \frac{\ln r}{1-r} - \max \ln|z_B|$

Summary

Phase space

- Physical states = Gaussian



- Distributions = Supernumbers

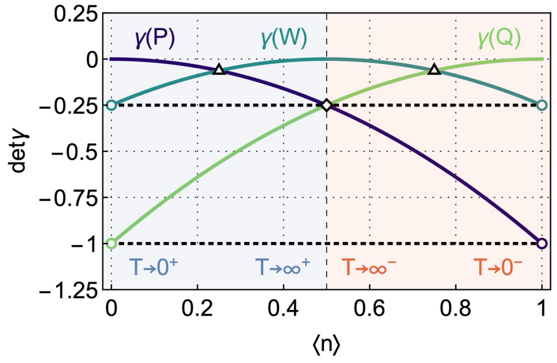
$$z(\alpha) = \begin{cases} -\langle n \rangle & \text{Glauber-}P \\ \frac{1}{2} - \langle n \rangle + \alpha\alpha^* & \text{Wigner-}W \\ 1 - \langle n \rangle & \text{Husimi-}Q \end{cases}$$

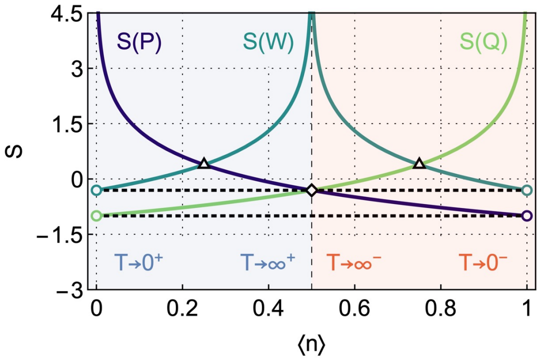
Information/majorization

- Majorization relations

$$z_1 \prec z \prec z_0$$

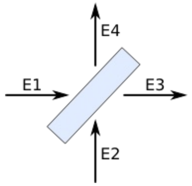
- Uncertainty relations

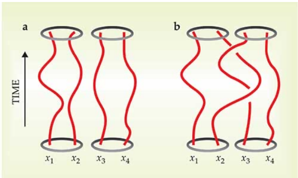




Outlook

- Quantum communication/computation with fermions?

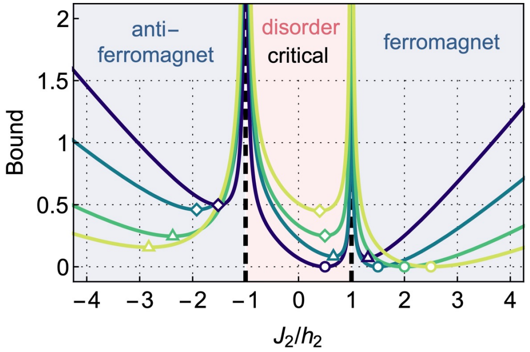




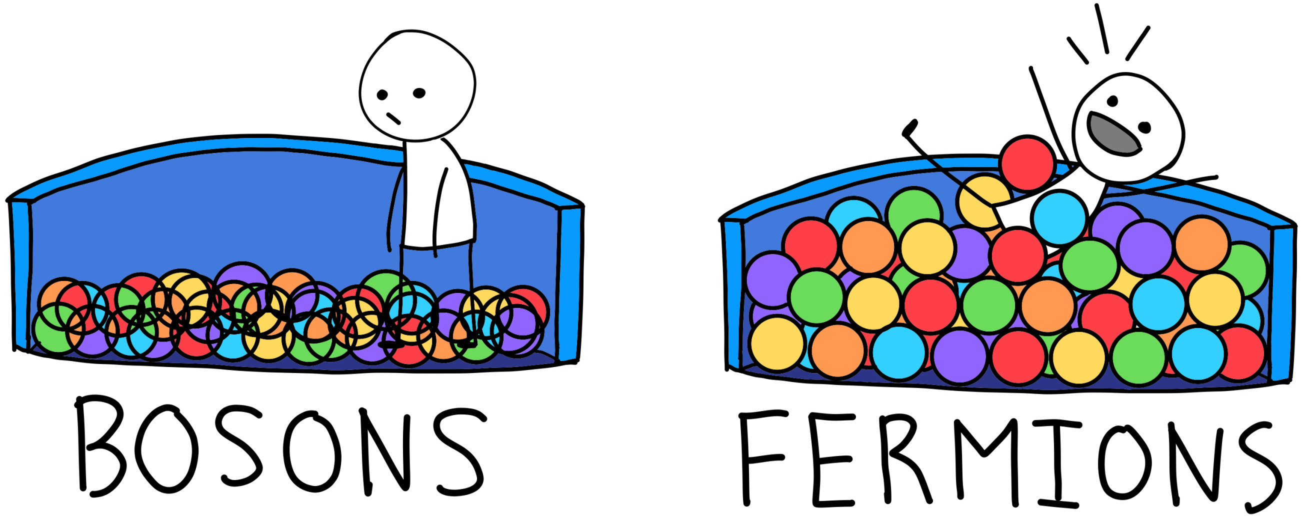
- Jordan-Wigner transform: fermion = spin-$\frac{1}{2}$ system

$$\boldsymbol{a}^\dagger = \tfrac{1}{2}(\boldsymbol{\sigma}_x + i\boldsymbol{\sigma}_y),\ \boldsymbol{a} = \tfrac{1}{2}(\boldsymbol{\sigma}_x - i\boldsymbol{\sigma}_y) \quad \rightarrow \text{understand Wigner negativity?}$$

- Quantum field theories → understand quantum phase transitions Ditsch '23



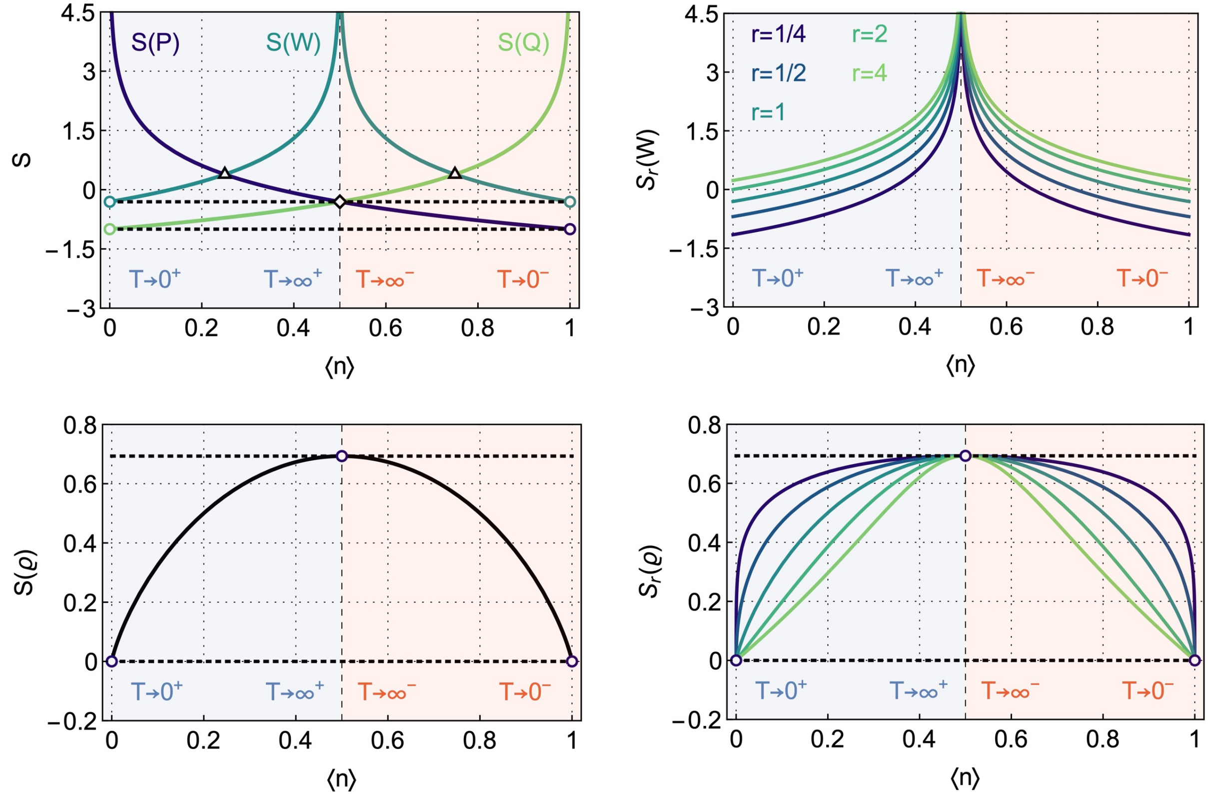
Thanks for your attention :)



Backup

Phase-space vs. quantum entropies

- Comparison



- convex
- possibly negative
- possibly infinite
- $\partial_r S_r(z) > 0$

- concave
- non-negative
- usually finite
- $\partial_r S_r(z) < 0$